The Search for Liquidity[*]

Jose A. Carrasco[†]   Lones Smith[‡]

Department of Economics

University of Wisconsin-Madison

November 18, 2015

Abstract

We formulate and solve a dynamic programming exercise that extends search theory to *perfectly divisible assets*, leading us to a theory of *search at the margin*. Our showcase application is the liquidation sale of a divisible asset — such as the time of an accountant, or a physical asset like recently inherited land. A seller faces a stream of buyers periodically arriving with random limit orders. His trading behavior optimally adjusts as the asset position falls, reflecting the endogenous time-varying value of the asset position. Our search theory story adds a different dimension to liquidity: the waiting time between trades.

We shift the focus from Bellman values to *Bellman value functions* in dynamic search. We recursively characterize three derivative properties of the Bellman value. First, even though dividend payoffs rise linearly in the position, so long as selling opportunities are individually bounded in size, there is "diminishing returns to optionality": The value function is strictly concave and thus the seller takes greater advantage of more generous offers, and his marginal value shifts up as he unwinds his position, making him less willing to trade. Second, if the density of purchase caps is falling, the marginal value of assets is strictly convex. Hence, the seller's supply response is less elastic at higher prices, and that the depth of the induced market rises in the asset position. The asset value function therefore shares standard properties of the utility for money: it is increasing, strictly concave, and the marginal value is strictly convex in the asset position.

Our model is amenable to price-quantity bargaining. We show that greater buyer bargaining power is tantamount to greater search frictions. Supply is higher and the negotiated prices lower with bargaining, and thus the induced market is more liquid.

[†]Web: www.tonocarrasco.com

[‡]Web: www.lonessmith.com

1 Introduction

Marginal analysis rules economic theory. By assuming perfectly divisible goods, choices are governed by first order conditions. Yet dynamic search theory has so far resisted such innovation, either focusing on the purchase or sale of finitely many units, or formally truncating individual search after one period. We formulate and fully solve a canonical and naturally-arising search problem for the sale of a large position of a *perfectly divisible illiquid asset*, resulting in a tractable fully dynamic theory of *search at the margin.*

We explore a dynamic programming exercise extending optimal dynamic search to the sale of perfectly divisible assets. We assume that an impatient owner of a large position in a divisible asset wishes to sell it off. The asset may involve human capital — like the time of a consultant, accountant, or other professional — or physical capital, like recently inherited land, or trees in a forest. The seller lacks an organized market, and instead can only slowly sell it off to a stream of randomly arriving buyers. Arrivals follow a stationary Poisson process, each with a limit order whose price and purchase cap are random. The accountant, for instance, may receive a retainer offer of up to three hours a week for $500 an hour; the forest owner may get an order of up to 1000 trees at $5 a tree. Until sale, he possibly also earns a flow dividend proportional to his holdings — the accountant may have a backup job; the land may be rented out for skeet-shooting. In each trading opportunity, either the buyer's cap limits the trade, or the seller declines the trade, or the seller partially exercises the buyer's offer. The trading schedule evolves in the falling asset position, reflecting the endogenous option value of a smaller position. This paper characterizes the nonstationary trading behavior as the seller unwinds his position. We then show that our analysis can easily incorporate negotiated prices, given by the Nash bargaining solution. This natural twist affects both the price and quantity.

A search for liquidity with indivisible units entails a 0-1 trading strategy for each unit — sell today or later. With a perfectly divisible asset, the marginal value governs his trading behavior, acting as the seller's marginal cost curve. When a buyer arrives, the seller trades off an immediate sure gain for an uncertain future one *for the marginal unit.*

Our Theorem 1 uses standard recursive methods to deduce that the Bellman value function is increasing in the asset position. But marginal analysis requires a differentiable value function. Our differentiability proof in Theorem 2 cannot appeal to Benveniste and Scheinkman (1979) since the optimal policy is frequently a corner solution. We instead establish the existence of a bounded derivative recursively. We show that if the value function is differentiable, then its derivative defines a contraction mapping that admits a unique bounded fixed point that is continuous. This new deduction method should prove

valuable in other contexts where corner solutions invalidate envelope theorem logic.

Of course, marginal analysis is only a valuable tool if second order conditions are met. Our next major result is a new "diminishing returns to optionality" property: *The Bellman value function is strictly concave in the asset position* (Theorem 3). The reason is that because the trading opportunities are bounded, the buyers' purchase caps bind more often with a larger position. Notably, the value function is strictly concave even though the seller's asset dividends rise linearly in his position. This might be the first dynamic programming problem in which strict concavity of the value is not inherited from the period payoff function — as it is in growth theory, for instance. We prove concavity this using recursive methods along with convex duality theory of Fenchel. We also show that the concavity attenuates as the position explodes, and eventually, he trades nearly myopically, only valuing an asset for its dividends (Lemma 2).

Diminishing returns to optionality has two key implications — that the seller takes greater advantage of more generous offers, and that his marginal value shifts up as he sells off his position, making him less willing to trade. For the value is lower but the marginal value is higher with a smaller position. The trading decision naturally follows. Because the marginal value is boundedly positive and boundedly finite, there is a *choke price*, below which the seller refuses to trade, and a higher *sell-all price.* Divisibility of the asset position therefore matters for intermediate prices, when the seller optimally partially unwinds his position. The explicit formula for supply in Corollary 1 is separable in the price and position, increasing in price and linearly increasing in position. While driven by convex costs in static economics, rising supply curves arise in our dynamic model due to diminishing returns to optionality. For an important application, in the 1990s, the Resolution Trust Corporation (RTC) was tasked with liquidating almost $400 billion of portfolios of 747 failed savings and loans in the S&L Crisis.[1] But the mandate of the RTC evolved through time: Originally it was required to secure 95% of market value; yet this fell in succession to 80%, 60%, and then 50%, opposite to our predictions.

We next exploit our recursion for the marginal value, proving that *the marginal value is not only positive and decreasing, but also convex in the position* (Theorem 4). For some intuition, assume that no offers were capped. In this case, the marginal value satisfies the same recursion as the value function does in wage search, and so would be a constant function. But with caps, the buyer might be unwilling to purchase that extra unit. We must therefore subtract the fraction of capped offers. This share is roughly the

---

[1] "I was on the oversight board of the RTC...Some of the stuff that the RTC wound up with was perfectly liquid and salable. But a big chunk was uncompleted eight-hole golf courses, half-built office towers, and vacant malls. Nobody wanted it. ...Needless to say, the bids were less than 50 percent of the original cost." — Alan Greenspan (Leonard and Coy, 2012).

cumulative distribution function of the offer caps, at that price. This fraction is concave when the density of offer caps is declining — a natural assumption that larger offers are decreasingly less likely. Since supply is inverse to the marginal value, when it is positive, the supply is a concave function of the price (Corollary 3). So while the seller purchases more from higher offers, his supply elasticity tapers off at larger asset positions.

In our paper, search frictions are captured by the buyer arrival rates and the seller's impatience. We predict how the value function and trading behavior change as either frictional measure — e.g. a thinner market, or holding more of a firesale. Intuitively, the search optionality is not as valuable when the frictions are greater. As a result, we use our recursive methodology to prove that the value and marginal value both fall, and less obviously so, the second derivative rises (Theorem 5). For an intuition, assume that the trading opportunities are rarer. Then the seller trades more in each, and so is less price sensitive; accordingly, his value and marginal value functions are both flatter.

Our search model offers richer insights on liquidity than typically pursued in finance, since we predict both trading behavior and the seller's optimal waiting time between sales. The seller faces a tradeoff between more frequent and predictable trades, and more profitable trades. But as the seller's asset position falls or frictions vanish our transactional liquidity measures worsen — namely, the depth in the induced asset market falls, and the premium over the choke price grows (Theorem 8). On the other hand, waiting times worsen as the asset position falls, and respond ambivalently to search frictions, falling with either a higher meeting rate or interest rate (Theorem 7).

Unlike settings with a continuum of buyers, buyers here enjoy market power. Our theory remains tractable and our results robust when Nash bargaining fixes the trade quantity and price. We uncover a general principle in §7 — that *greater bargaining power for buyers is formally equivalent to increased search frictions*: It raises the supply, reduces the negotiated price, improving market liquidity. All told, the price and quantity move oppositely. With bargaining, for instance, the supply still rises in the position, but the negotiated price falls. As an application, assume that buyers A and B have the same reservation price, but that B wishes to acquire a larger share of the seller's position. Then B will pay a higher negotiated price — he will not get a quantity discount.

1. LINK TO MONETARY SEARCH THEORY. Trade models are the primary potential application of search at the margin, since goods, assets, and money are best modeled as perfectly divisible.[2] The literature is too vast for us to survey, but these models all

---

[2]Smith (1994) explores how search frictions affect trade with indivisible heterogeneous goods and unit demands where "beauty is in the eye of the beholder". Krainer and LeRoy (2002) also explore a search and matching equilibrium model of illiquid indivisible asset valuation.

embody indivisibilities explicitly or implicitly that escape solving a dynamic search exercise with a perfectly divisible asset. Monetary theory is the largest body of work here — Lagos, Rocheteau, and Wright (2015) call these "New Monetarists". Kiyotaki and Wright (1989), the first generation of this work, assumed pairwise trade of indivisible commodities that act as money. Later, work spawned by Green and Zhou (1998) allowed for perfectly divisible money, but assumed a finite mesh for money holdings; they also assume single unit trades. Lagos and Wright (2005) delivers a tractable monetary model with divisible assets and a divisible good. But there, search takes one period only (the "day"), and then is followed by an unchanging frictionless centralized "night" market that fixes the outside option value of money. That periodic access to a market is itself a search friction underlies Lagos and Rocheteau (2009) too, who assume asset divisibility in a search equilibrium model of financial intermediation in an over-the-counter market. Their investors randomly meet dealers (all are identical), who trade for them in a competitive market. This renders the outside option value of assets constant, and thus the investor's selling strategy depends forwardly on the expected market price, but is *independent* of his asset position — the essence of our search at the margin. A stochastically stationary decision model also underlies Lagos, Rocheteau, and Weill (2011).

2. LINK TO FINANCIAL MARKET MICROSTRUCTURE. An instructive contrast to our paper is the financial market microstructure literature. For instance, Kyle (1985) explores trade by a large insider, with private information about the asset value. Like us, he explores how liquidity is endogenously fixed in a dynamic environment. His insider trades into a specialist market, whereas ours searches for counterparties. He assumes a diffusion counterparty arrival process, rather than our periodic Poisson arrivals. To deal with the informational asymmetry, he must solve a dynamic sequential equilibrium, whereas ours need only solve a forward-looking decision problem. This insider *depresses the price by selling more* (out of equilibrium), but *our derived supply curve is increasing.* This insight is opposite to our Corollary 1, and highlights the importance of asymmetric information in trade. Depth is infinite in a standard competitive equilibrium, and is finite in Kyle due to the informational asymmetry and the arrival process, while it is finite in our setting simply because of the arrival process. In fact, depth in Kyle (1985) is constant, while our depth monotonically falls in time as the insider unwinds his position.

Since our paper lacks a close prequel, we next relate our theory to the McCall wage search model in §2. We give the model in §3, characterize the value function and supply schedule in §4, and do sensitivity analysis in §5. We analyze liquidity measures in §6, and introduce bargaining in §7. We highlight the recursive logic throughout. Gentle and instructive proofs are in the text, and lengthier ones postponed to the appendix.
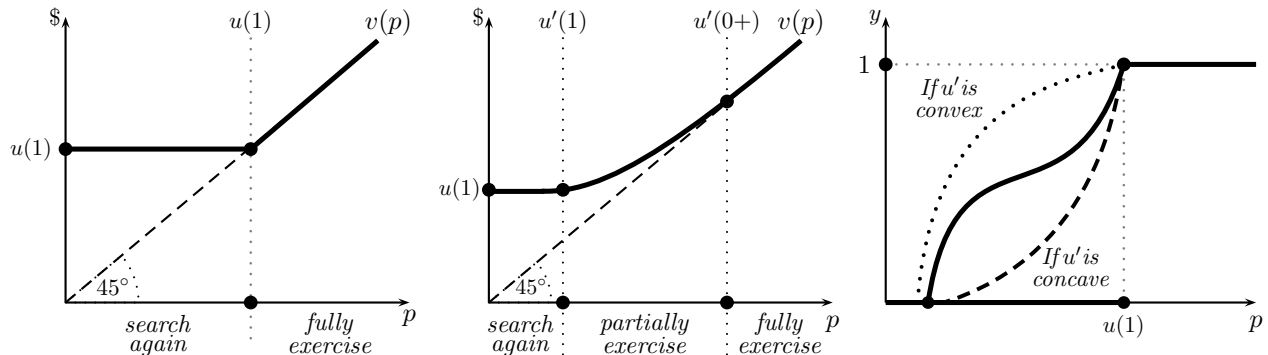
4

Figure 1: **Search with Indivisible versus Perfectly Divisible Units.** The left two panels depict wage search, and a convex payoff value $v(p)$. The shape of the optimal supply $y^*(p)$ (right panel) requires a characterization of the marginal continuation value.

2 Search at the Margin: A Foretaste

To fix ideas, simply consider a two period model without discounting, and take as given the continuation value function u. In the first period, a seller holds a unit of an asset and meets a buyer. He chooses how much $y \geq 0$ to sell at the random offered price $p > 0$. In period two, he derives utility $u(1-y)$. We assume that $u(0) = 0$ and that u is increasing. Optimal trade behavior yields the ex ante value $v(p) = \max_{y \in [0,1]}(py + u(1-y))$.

Assume first that $u(a)$ is linear or convex. Then $v(p) = \max\{p, u(1)\}$, as in the left panel of Figure 1. The outside option is worth $u(1)$ and the inside option p. *In a the stationary wage search model, $u(1)$ is the reservation wage and p the current wage offer* (McCall, 1970). The seller fully exercises his option when $p \geq u(1)$; otherwise, he sells nothing. So the divisibility assumption is irrelevant: the same outcome arises when assets are both indivisible ($y = 0, 1$) or divisible ($y \in [0,1]$).

Assume next that the continuation value $u(a)$ is strictly concave and differentiable on $[0,1]$. As seen in Figure 1, he sells nothing if $p < u'(1)$, and sells out for all high prices $p \geq u'(0+)$. For intermediate prices $u'(1) \leq p < u'(0+)$, he partially sells his position, and the FOC $p - u'(1-y) = 0$ fixes the optimal supply $y^*(p) = 1 - (u')^{-1}(p)$. So strict concavity of $u(a)$ yields a positive and increasing supply (the solid line in Figure 1, at right). Since the slope of the ex ante value $v(p)$ is the sales $y(p)$, to fully understand the trading behavior requires a specific characterization of the marginal continuation value.

This paper's contribution is thus to *shift focus from Bellman values to Bellman value functions in dynamic search*. Our efforts are directed at recursively proving that — even though dividend payoffs are linear in the asset position — the utility function u is differentiable and strictly concave, and the marginal utility u' strictly convex. We also derive "comparative statics" for the value function and resulting trading behavior.

5

3 The Model

Time is continuous on $[0, \infty)$. An infinitely-lived *seller* owns a large *position* $a < \infty$ of a perfectly divisible asset. Each asset share pays him a constant dividend $0 \leq k < \infty$ per unit time, which he discounts at the interest rate $r > 0$.

The asset lacks a formal market. The seller, however, periodically meets a buyer, each arriving at a random time with a random offer. Arrivals T follow a Poisson process with *arrival rate* $\rho > 0$. In a meeting, the buyer makes a limit order *offer* (p, x) specifying the share *(bid) price* $p > 0$ and *purchase cap* $x > 0$, i.e. the maximum desired quantity. The offer dimensions (P, X) are possibly dependent random variables, with finite mean, from a bounded continuous density $f(p, x)$, weakly falling in x. Denote the cdf by $F(p, x)$, expectations w.r.t. f by $\mathbb{E}$, and marginals by $g(p), h(x) > 0$ on $(0, \infty)$, with $g(p)$ log-concave. The marginal $h(x)$ and conditional expected price $\mathbb{E}[P|x]$ are uniformly bounded, i.e., $h(x) \leq \bar{h} < \infty$, and $\mathbb{E}[P|x] \leq \bar{p} < \infty$ for any $x > 0$.

Our paper simultaneously solves for two possible versions of the model. In the simplest case, the buyer's offer is take-it-or-leave-it. After receiving an offer (p, x), the seller chooses how much to sell $y \in [0, \min\{x, a\}]$. For instance, he cannot short sell.

Alternatively, the buyer is described by a *reservation price* $w > 0$ and a *purchase cap* $x > 0$, with density $f(w, x)$. But now the *terms of trade* — price and quantity (p, y) — arise from the Nash bargaining solution. We precisely specify this variation in §7.

After selling quantity $y \leq a$ at price p, the seller continues his search with new position $a - y$, and a cash inflow of ay. *He seeks to maximize his expected present value of cash flows from dividends and sales.* We let $V(a)$ be the value of position $a \geq 0$.

4 The Value Function and Selling Strategy

When meeting a buyer, the seller optimally decides whether and how much to avail himself of the proposed terms of trade. In so doing, he trades off a sure immediate gain for the option value of future trades. Since one available policy is never to trade, we must have $V(a) \geq ak/r$. But the right side is an unbounded function when $k > 0$, and so we instead focus on the net-of-dividend *option value function* $W(a) = V(a) - ak/r \geq 0$.

We now solve the problem recursively, and characterize its solution using dynamic programming for the state variable that is the asset position $a \geq 0$. Since $\mathbb{E}_T \int_0^T e^{-rs} ds = \rho/(r + \rho)$, the Bellman equation for V yields the option value recursion:

$$W(a) = (\mathcal{T}W)(a) = \frac{\rho}{r + \rho} \mathbb{E}\left(\max_{y \in [0, \min\{X, a\}]} \left[\left(P - \frac{k}{r} \right) y + W(a - y) \right] \right) \tag{1}$$

Namely, upon meeting a buyer with offer (p, x), the seller maximizes the present value $py + V(a - y) = py + W(a - y) + (a - y)k/r = (p - k/r)y + W(a - y) + ak/r$ over sales y.

The seller chooses how much of the limit order offer (p, x) to supply. We call the solution the *supply function* $\mathcal{Y}(p, x, a)$, and prove that it is uniquely defined below.

Imagine instead that the seller chooses the new position $a' = a - y$, and recast her optimization (1) as $(p - k/r)a - \min_{[a - \min\{x, a\}, a]}[pa' - V(a')]$, namely, minimizing the opportunity cost $pa' - V(a')$ of holding assets. We exploit this transformation below.

To fix ideas, consider two extreme cases. If the seller has no option to sell ($\rho = 0$), his value reduces to the discounted value of dividends $V(a) = ak/r$. Also, when the asset pays no dividends ($k = 0$), the value is a pure option on randomly meeting buyers, whose proposed terms of trade are acceptable. The right hand side of (1) includes a positive probability that the offer is unacceptable, whence the seller offers zero supply.

Let $\mathcal{B}$ be the space of bounded continuous functions on $[0, \infty)$ with the sup norm.

Lemma 1 $\mathcal{T}$ *is a contraction with a unique bounded and continuous fixed point W in $\mathcal{B}$.*

We use a natural upper bound on the option value function W: Even given an infinite position $a = \infty$, when one fully exploits all offers, one still only secures a finite present value $\rho \mathbb{E}[PX]/r < \infty$. The proof in the appendix shows that this upper bound for $W(a)$ is preserved, and then applies Blackwell's sufficient condition for a contraction.

We now recursively derive all properties of the value function.

Theorem 1 *The value function $V(a)$ is a strictly increasing and concave function of a.*

Proof: Since $\mathcal{T}$ is a contraction, and concavity is a closed property in $\mathcal{B}$ with the sup norm, it suffices that $\mathcal{T}$ preserves concavity (Corollary 3.2.1 in Lucas, Stokey, and Prescott (1989)). Let the seller choose the post trade asset position $z \equiv a - y$ from the convex constraint set $C(x) = \cup_a \{(z, a) | \max\{a - x, 0\} \le z \le a\}$. To eliminate the constraint, we introduce the characteristic function $\chi_{C(x)}(z, a) = 0$ if $(z, a) \in C(x)$ and $+\infty$ otherwise. So $\chi_{C(x)}$ is convex since $C(x)$ is convex. Rewrite expression (1) as:

$$(\mathcal{T}W)(a) = \frac{\rho}{r + \rho} \left(a \left(\mathbb{E}[P] - k/r \right) - \mathbb{E} \left(\min_{z \ge 0} [(P - k/r)z - W(z) + \chi_{C(x)}(z, a)] \right) \right) \quad (2)$$

For the recursion, assume that W is concave. Then $pz - W(z) + \chi_{C(x)}(z, a)$ is convex in (z, a). By Theorem 5.3 of Rockafellar (1970), $\min_{z \ge 0}[pz - W(z) + \chi_{C(x)}(z, a)]$ is convex in a. Since expectation preserves concavity, $(\mathcal{T}W)(a)$ is concave in a, and so too is its fixed point $\mathcal{T}W = W$. Finally, as the value of dividends ak/r is linear, V is concave. $\square$

To see that the concavity logic is unrelated to standard duality theory in economic theory, observe that *a minimization* (2) *yields a convex objective.* Or equivalently, our primary option value *maximization* (1) engenders a *concave* value function W.

Wage search models solve a pure stopping exercise (left panel of Figure 1), requiring a reservation wage or price. Our seller's supply function is continuously variable. As seen in §2, the solution requires characterizing the marginal value function. For a first step, we next prove that the marginal value function finitely exists.[3] The value V is well-defined since $V(a) = W(a) + ak/r$. We now proceed recursively. When V' exists, we show in §A.3 that it solves $V' = \mathcal{S}V'$, where we compactly write $\mathcal{S}V'$ as:

$$(\mathcal{S}V')(a) = \frac{k}{r+\rho} + \frac{\rho}{r+\rho}\mathbb{E}\left(\max\left\{V'(a), \min\left\{P, V'(a - \min\{X, a\})(1 + \chi_{[0,a]}(X))\right\}\right\}\right) \ (3)$$

We prove in §A.3 that $\mathcal{S}$ is a contraction on $\mathcal{B}$, and its unique fixed point in $\mathcal{B}$ is V'.

Theorem 2 *The marginal value $V'(a)$ exists on $[0, \infty)$, is continuous, and exceeds k/r.*

The proof finds an upper bound $V'(a) \leq \rho\mathbb{E}[P]/r < \infty$. We now strengthen Theorem 1.

Theorem 3 *The value function $V(a)$ is a strictly concave function of the position a.*

The proof in §A.4 first argues directly that V cannot be linear on any interval $[0, a]$, for the optimal policy would then entail a constant choke price equal to $V'(0+)$; however, that policy would induce a strictly concave value given the purchase caps. The proof then extends this logic to preclude linearity of the value function on any interval.

This result highlights the critical role played by the purchase caps. For absent any caps, the marginal value recursion (3) reduces to the standard Bellman equation for wage search, namely, $rV'(a) = k + \rho\mathbb{E}(\max\{P - V'(a), 0\})$. In this case, the value function is linear $V(a) = aV'(0+)$, and the option to partially unwind the position is worthless.

Because the asset position a confers valuable trade opportunities, its value strictly exceeds the present value ak/r of its dividend stream. In fact, the concavity in Theorem 3 follows from the bounded limit orders. For consider the extreme case of a zero dividend. Here, the value just owes to the sales option. Since a larger position take more time to unwind, incremental assets are sold farther in the future, and the marginal value falls.

---

[3]A concave function is almost everywhere differentiable. We argue that it is everywhere differentiable. Benveniste and Scheinkman (1979) prove this by restricting primitives of the problem so that the solution is in the interior of the choice set. Rincon-Zapatero and Santos (2012) dispense with this interiority assumption, only requiring the existence of an interior optimal path. Aliprantis, Camera, and Ruscitti (2009) and Rincon-Zapatero and Santos (2009) explore this differentiability in discrete times problems. In both papers, assumptions that guarantee existence of an interior path are made. Our proof only requires bounded and continuous densities and finite expected price.
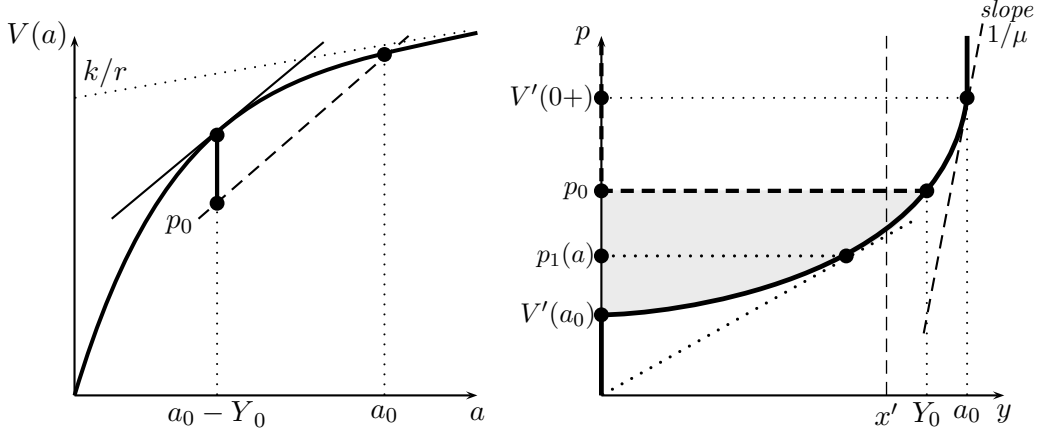
Figure 2: **The value function and inverse uncapped supply**. At left is the increasing and concave value function $V(a)$. Given a limit order price p_0, the trade surplus is the peak vertical distance of $V(a)$ from the dashed line with slope p_0. At right, the seller supplies the lesser of the purchase cap x and Y_0. For any cap x', the supply is truncated and rises vertically. This optimal supply for the seller maximizes his "producer surplus", and equates price and the value of the marginal unit $V'(a - y)$. Also, there is a unique secant tangent to Y at some price $p_1(a) > 0$. By Corollary 3, the supply elasticity exceeds one and is falling for $p < p_1(a)$, and otherwise is less than one.

Waiting until a buyer arrives willing to purchase everything is intuitively suboptimal. But how should the seller react to the partial purchase offers? With the value function, this reduces to an easy static sales exercise, in which the marginal cost function is the marginal value function, as in §2. The seller's supply is the inverse marginal value function $(V')^{-1}$, but now the purchase cap may bind on his desired supply (Figure 2).

Corollary 1 *For any asset position $a > 0$, share price p, and cap x, the optimal supply is $\mathcal{Y}(p, x, a) = \min\{x, Y(p, a)\}$, where the* uncapped supply $Y(p, a)$ *is given by:*

$$Y(p, a) = \begin{cases} \max\{a - (V')^{-1}(p), 0\} & for \quad p \le V'(0+) \\ a & for \quad p > V'(0+) \end{cases} \tag{4}$$

Proof: In trade meetings, the seller solves $\max_y [py + V(a - y)]$ s.t. $0 \le y \le x$ and $y \le a$. As V is strictly concave and the constraints are linear, the FOC is necessary and sufficient for a maximum. Since at most one constraint binds, the constraint qualification for the Kuhn-Tucker conditions is met. If the multipliers are respectively $\lambda_1, \lambda_2, \lambda_3 \ge 0$, then the FOC is $p - V'(a - y) = -\lambda_1 + \lambda_2 + \lambda_3$. By complementary slackness, (*i*) if $y = x \le a$, then $p - V'(a - x) \ge 0$, and (*ii*) if $y = a < x$, then $p - V'(0+) \ge 0$, and (*iii*) if $y = 0$, then $p - V'(a) \le 0$. Otherwise, all multipliers vanish, and $p = V'(a - y)$. $\square$

Notice that never trading is not optimal since this would yield payoff $ak/r < V(a)$, contrary to Theorem 2. Rather the supply (4) is sometimes positive. For any position $a < (V')^{-1}(p)$, the supply function vanishes, but thereafter, it rises dollar for dollar in the position, until it hits the cap x. Equally well, as a function of the purchase cap, supply is either zero (at low prices), or increasing dollar for dollar in the cap until hitting the *uncapped supply* $a - (V')^{-1}(p)$. Next, consider supply as a function of the bid price. Below the *choke price* $V'(a)$, supply vanishes. For higher bid prices that are below the *ask price* $V'(a-x)$, offers are only partially acted upon. For higher bid prices, offers are fully acted upon. Above the *sell-all price* $V'(0+)$, any offer is fully acted upon at *any* position. Curiously, from (3), $V'(0+)$ solves the same Bellman equation as does the value function in a "parallel wage search problem", namely, $rV'(0+) = k + \rho\mathbb{E}[\max\{P - V'(0+), 0\}]$. So standard search theory mimics ours with divisible assets and a small position (Figure 3).

With indivisible supply, standard search theory would dictate a choke-price below which no sale occurs. But the optimal sales policy is not constant in time here. Rather, the choke-price reflects the *marginal* value, and the supply is a variable intensive margin that adjusts in the position. From strict concavity, the seller expects a *trade surplus*:

$$\sigma(a) = \mathbb{E}[P\mathcal{Y}(P,X,a) + V(a - \mathcal{Y}(P,X,a)) - V(a)] = \mathbb{E}\max_{y \in [0,\min\{X,a\}]} \int_0^y [P - V'(a-t))]dt \quad (5)$$

We can suggestively rewrite (1) as $W(a) = \rho\sigma(a)/r > 0$, reflecting how *the option value is the present value of trade surplus*. The associated Bellman equation for the Bellman value V is thus:

$$rV(a) = ak + \rho\sigma(a) \quad (6)$$

Differentiating (6) then yields $rV'(a) - k = \rho\sigma'(a)$, and so $\sigma'(a) > 0$, since $V'(a) > k/r$ by Theorem 2. That $\sigma(a)$ is strictly concave follows from (6) and Theorem 3. All told, *the trade surplus $\sigma(a)$ is increasing and strictly concave in the asset position*. We now ask what fraction of the Bellman value reflects the anticipation of future trade surplus.

Lemma 2 *The search optionality share of value $\sigma(a)/V(a)$ falls in the asset position a, and vanishes as $a \to \infty$ if $k > 0$. Thus, the limit marginal value is $\lim_{a \to \infty} V'(a) = k/r$.*

Proof: First, (6) implies $\rho\sigma(a)/V(a) = r - ak/V(a)$. But the secant slope $V(a)/a$ falls in a, as V is increasing and concave, with $V(0) = 0$. Hence, $\sigma(a)/V(a)$ falls in a. Finally, $V'(a) \to k/r$ iff $V(a)/a \to k/r$ by L'Hôpital, and by (6), iff $\sigma(a)/V(a) \to 0$ which is iff $\sigma(a)/a \to 0$. We prove this last limit. The trade surplus obeys $py + V(a-y) - V(a) \leq py$. Maximizing over $y \leq \min(x,a)$, and taking expectations in P, X, yields an upper bound

$\sigma(a) \leq \mathbb{E}(PX)$. Finally, let $a \to \infty$ in $0 \leq \sigma(a)/a \leq \mathbb{E}(PX)/a$. □

So the seller increasingly ignores search optionality as his asset position grows, and his asset value eventually only reflects its dividend value to him — e.g., his marginal and average values $V'(a)$ and $V(a)/a$ both converge upon k/r as $a \to \infty$. Trading behavior is likewise asymptotically stationary, and k/r is both the limit choke price and sell-all price. In other words, the optimal supply becomes infinitely elastic near the price k/r.

Theorem 4 *The marginal value $V'(a)$ is decreasing and strictly convex in assets a. Moreover, $V''(a) < 0$ exists on $(0, \infty)$, is continuous, and is at least $-(\rho/r)\bar{p}\bar{h}$.*

Notably, the value function of assets has the same properties typically assumed for utility functions for money u: increasing, risk averse, and prudent ($u'' < 0 < u'$ with u' convex).

Let us see why the driving force behind Theorem 4 is the falling purchase cap density. Note that the seller requires an optimal trading policy for each asset unit. First of all, suppose the seller sells an extra $da = 1$ asset units *irrespective of the offer price*. This yields payoff $rV'(a) = k + \rho\mathbb{E}[P - V'(a)]$, by (6). Improve on this using the logic of indivisible assets, imposing a reservation price $V'(a)$. This yields the recursion $rV'(a) = k + \rho\mathbb{E}[\max\{P - V'(a), 0\}]$, with a larger marginal value. But then the marginal value is constant, contrary to Theorem 1. The reason is that it ignores feasibility, as the buyer may be capped out. Consider now the special case when price and quantity are independent, or $f(p, x) = g(p)h(x)$. Since the buyer is capped out with probability $H(Y(p, a))$, the seller suffers an expected loss $pH(Y(p, a))$, but can retain his marginal assets. With a decreasing density H', the expected loss is concave in a; our proof in §A.5 argues that this concave subtraction from the marginal value leads to convexity.

Given the uncapped supply $Y(p, a) = a - (V')^{-1}(p)$ by Corollary 1, we conclude:

Corollary 2 *When positive, supply $Y(p, a)$ is increasing and concave in p.*

For since V' is decreasing and convex (Theorem 4), its inverse is decreasing and convex in p, and so $Y(p, a)$ is increasing and concave in p.

The expression (4) for uncapped supply reveals that the seller grows more picky and trades less as he unwinds his position. Yet at the same time, we next argue that the seller grows more price sensitive. For consider the *supply elasticity* $\eta(p, a) \equiv Y_1(p, a)p/Y(p, a)$. The convex marginal value sheds light on its slope via the second derivative of the value function. Indeed, since supply $Y(p, a)$ is concave in p, it is least elastic at the sell-all price $p = V'(0+)$, with limit slope $\mu = Y_1(V'(0+)-, a) < \infty$. We also call $p_1(a)$ the price that maximizes the slope of the secant $Y(p, a)/p$ in $[0, V'(0+)]$, as seen in Figure 2.
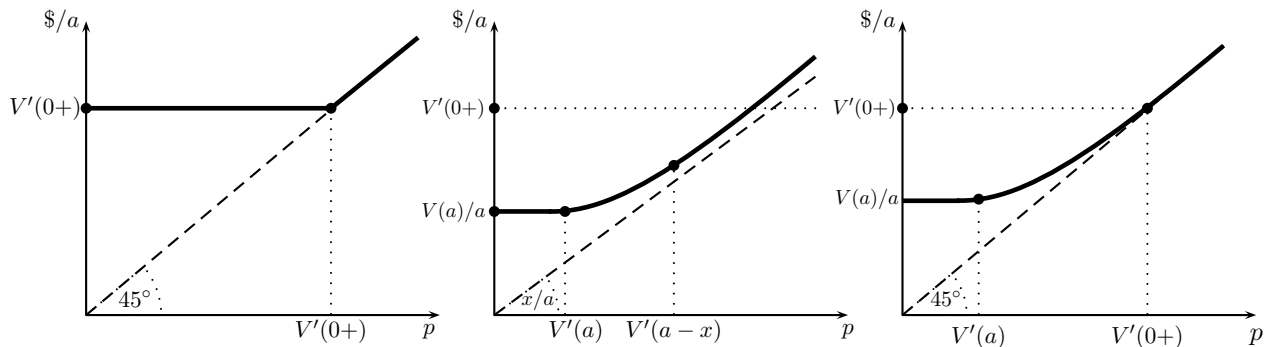
Figure 3: **The Value of Divisibility in Search**. This modifies Figure 1, adding purchase caps. The asset position is indivisible at left. At the middle and right are the cases with divisible assets for prices $p < V'(0+)$. The position is a perpetual call option with zero strike price. The panels plot two functions of the price: the Bellman value per unit of asset (thick lines), i.e., $[p\mathcal{Y}(p,x,a) + V(a - \mathcal{Y}(p,x,a))]/a$, and the intrinsic value of the option (dashed lines): $p\min\{x,a\}/a$.

Corollary 3 *The supply elasticity $\eta(p,a)$ exceeds one for low asset positions or prices. Specifically, the supply elasticity is $\eta(p,a) = 0$ for $p \geq V'(0+)$, but exceeds one for $a \leq \mu V'(0+)$, or for $a > \mu V'(0+)$ but low enough prices $p < p_1(a)$.*

Altogether, with wage search, a trader stops when he secures a wage in excess of the reservation wage. But in our setting with divisibility and strict concavity, the average and marginal values diverge, and the seller's trading strategy is governed by the marginal value. Trade may be choked off by the buyer or the seller (middle and right panels of Figure 3, respectively). Given the concavity, the average always exceeds the margin.

5   Changing Search Frictions and Offer Distributions

With a zero flow dividend, the search friction measure $\psi = r/\rho$ is the only model parameter. More generally, behavior reflects the triple (k, ρ, r). We now characterize how these model parameters influence the derived functions V, V', V''. While the value function obeys $V' > 0 > V''$, we now argue that each inequality grows stricter as search frictions diminish: the marginal value naturally rises, but the second derivative falls. Our recursive proof exploits a key lemma in Albrecht, Holmlund, and Lang (1991).

Theorem 5 *For any asset position $a > 0$, the value $V(a)$ and marginal value $V'(a)$ fall in r and rise in ρ and k, while $V''(a)$ rises in r and falls in ρ and k.*

The comparative statics of V and V' parallel those in the stationary indivisible search model (Figure 4): As search frictions fall, the value and marginal value increase. But

the marginal value falls faster at lower frictions. To wit, the value function flattens.

Since trade surplus $\sigma(a)$ is the maximal area (5) over the marginal value and below the price, it rises in r and falls in ρ and k. Since $rV(a) = ak + \rho\sigma(a)$ by (6), we see that while the value $V(a)$ falls in r, it is less than unit elastic, since $rV(a)$ rises. Just the same, while trade surplus $\sigma(a)$ falls in ρ, it is less than unit elastic since $V(a) = ak/r + \rho\sigma(a)/r$ rises in ρ, by Theorem 5. Finally, the value $V(a)$ rises in the dividend k, but is less than unit elastic since $\sigma(a)$ falls in k, given that $V'(a)$ falls, by Theorem 5. All told, the share of the value reflecting the search optionality $\rho\sigma(a)/(rV(a)) = 1 - ak/(rV(a))$ falls in the dividend k, but rises in the interest rate r and arrival rate ρ. Counterintuitively, *search optionality proportionately matters more with greater impatience, and so asset values proportionately reflect dividends more at lower interest rates*. This paradox arises with indivisible assets since the value V and marginal value V' co-move in Theorem 5, and surplus σ moves inversely to V'.

Next, we explore how shifts in the offer distributions affect the value. We consider changes in the price distribution P conditional on a quantity, fixing the purchase cap marginal $h(x)$, and in the quantity distribution X, conditional on a price, fixing the price marginal $g(p)$. We call these *conditional* stochastic dominance relations.

Theorem 6 *The value V rises with (i) conditional first order stochastic dominance increases in P or X, (ii) conditional mean-preserving spreads in P, or (iii) conditional second order stochastic order increases in X. The marginal value V' rises with (i).*

As with search theory with indivisible units, like wage search, the seller profits from stochastically better or riskier wages. But in our model, the seller is now *harmed by quantity risk*. Since offers are truncated by his position, he exploits only the lower tail of the purchase cap distribution. In the proof below, we exploit Fenchel duality to show that the purchase caps induce a concave optimized trade payoff for every price.[4]

Proof of Theorem 6: Rewrite the *trade payoff* in (1) as $\max_y[(p - k/r)y + W(a - y) - \chi_{[0,\min\{x,a\}]}(y)]$. First, consider price changes. The maximum increases in p. Since it is the conjugate function of $(k/r)y - W(a - y) + \chi_{[0,\min\{x,a\}]}(y)$, it is convex in p, by Theorem 12.2 of Rockafellar (1970). If, for all x, the price distribution $\tilde{P}$ dominates P by first order stochastic dominance, or if $\tilde{P}$ is a mean preserving spread of P, then by

---

[4]It is natural to ponder the effect of risk on the marginal value, and thereby on trading behavior. All our numerical simulations suggest that V' falls with mean preserving spreads in the purchase caps X. We have been unable to formally establish this recursively, since the right side of (3) is not concave.
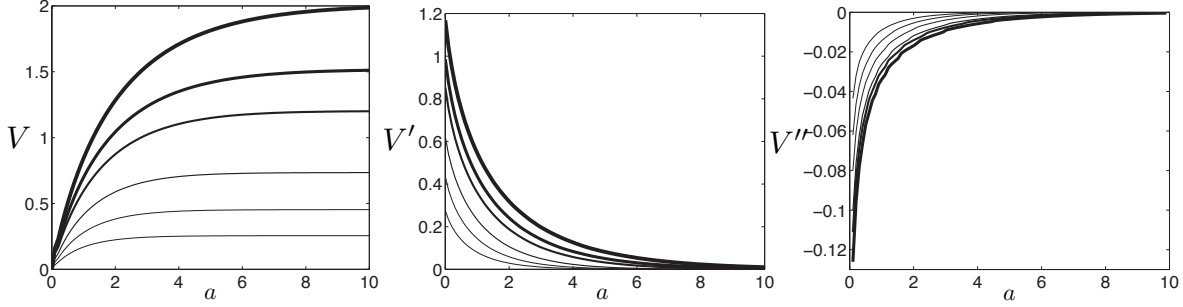
Figure 4: **How Search Frictions Affect $V(a), V'(a), V''(a)$.** We assume no dividends, so that only the value function only depends on $\psi = r/\rho$. From thick to thin lines, we plot numerical dynamic programming simulations as search frictions ψ increase: 0.005, 0.1, 0.2, 0.5, 1 and 2. We posit $P \sim \Gamma(1,1)$ and $X \sim \Gamma(0.5,1)$, independently.

basic stochastic dominance ranking theorems:

$$\mathbb{E}\left(\max_{y \in [0,\min\{x,a\}]}\left(\left(\tilde{P} - \frac{k}{r}\right)y + W(a-y)\right)\right) \geq \mathbb{E}\left(\max_{y \in [0,\min\{x,a\}]}\left(\left(P - \frac{k}{r}\right)y + W(a-y)\right)\right) \quad (7)$$

Define the Bellman operator $\tilde{\mathcal{T}}$ for $(\tilde{P}, X)$. Since the x-marginals coincide $\tilde{h}(x) = h(x)$, the expectation of (7) in X yields $(\tilde{\mathcal{T}}W)(a) \geq (\mathcal{T}W)(a)$, i.e. the fixed points obey $\tilde{W} \geq W$.

We now use a similar logic and consider cap changes. Since the constraint set $0 \leq y \leq \min\{x,a\}$ is convex in (y,x), so is the characteristic function $\chi_{[0,\min\{x,a\}]}(y)$. As a result, the trade payoff is concave in x, as in the proof of Theorem 1, by concavity of W. Hence, by ranking theorems, the fixed point W rises in FSD and SSD shifts in X.

Lastly, $\max\{V'(a), \min\{p, V'(a - \min\{x,a\})(1 + \chi_{[0,a]}(x))\}\}$ in (3) rises in p and x. So its expectation V' rises with first order stochastic dominance increases in P, X. $\quad\square$

6 Trading Behavior and Liquidity Measures

Broadly speaking, liquidity is a measure of the ease of trade. It reflects exogenous restrictions faced by the seller — here, the meeting rate and the offer distributions. When either rises, the market is more liquid. But a true measure of liquidity should reflect the seller's behavior, since everything can be sold at a zero price. For that reason, we develop some meaningful endogenous liquidity measures, assuming optimal behavior.

If a buyer offers more generous terms of trade, the seller is willing to part with more of his position (Figure 2). This *transactional liquidity* reflects the seller's forward-looking and nonstationary behavior as he trades off sure money today and possible money tomorrow. But given the exponential arrivals, our search model affords a second liquidity measure: The arrival rate of desirable offers is endogenous and equals $\rho\Phi(a) =$
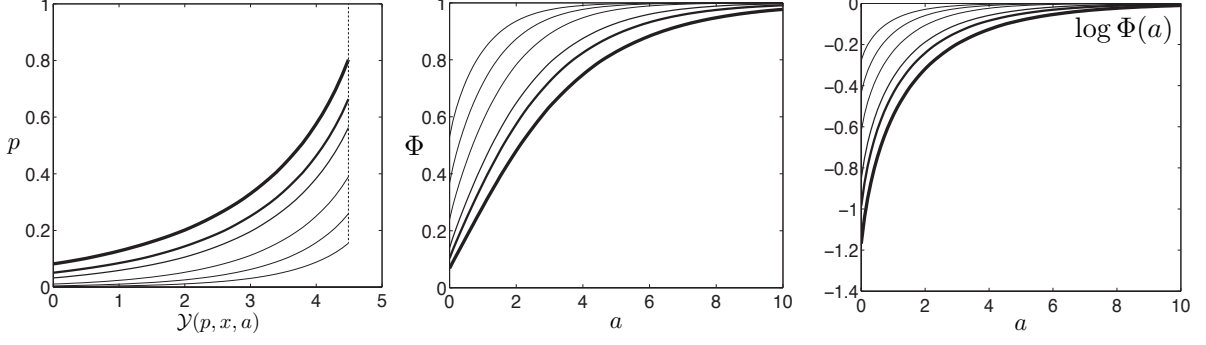
14

Figure 5: **How Search Frictions Affect Supply and the Trade Chance**. We plot numerical dynamic programming simulations for $P \sim \Gamma(1,1)$ and $X \sim \Gamma(0.5,1)$ and $k = 0$. Left: The supply for $a = 5$ and $x = 4.4$. Center: The trade chance as a function of the position. Right: The log of the trade chance as a function of assets. In all cases, frictions $\psi = r/\rho$ increase from thick to thin lines, $\psi = 0.005, 0.1, 0.2, 0.5, 1$ and 2.

$\rho(1 - F(V'(a), \infty))$, where $\Phi(a)$ is the *trade chance*. The expected time to trade is therefore $\tau(a) = 1/(\rho\Phi(a))$ and its variance $\xi(a) = 1/(\rho\Phi(a))^2$. Since both fall in the trade chance, with a smaller position, higher dividends or a lower interest rate, the seller is less eager to sell, and the mean and variance of trade times accordingly increase. But with a higher arrival rate ρ, a tradeoff emerges: The seller can afford to hold out for better offers, but they come more often. Given the log-concave price distribution, the first effect does not overwhelm the second, at least for small ρ.

Theorem 7 *The expected time to trade $\tau(a)$ and its variance $\xi(a)$ are decreasing and log-convex in the position a. When $\rho \leq \bar{\rho}$, both τ and ξ fall in r, rise in k, and fall in ρ.*

Proof: So it suffices that $\Phi(a)$ increase in r, decrease in k, and be increasing and log-concave in a. Now, $V'(a)$ falls in a by Theorem 3, and falls in r and increases in k by Theorem 5. So the trade chance $\Phi(a) = 1 - F(V'(a), \infty)$ increases in a and r, but falls in k. Since the epigraph $\{(p, a)|p \geq V'(a)\}$ of the convex function V' is convex, its indicator function is log-concave. Because the marginal $g(p)$ is assumed log-concave, the trade chance written as $\Phi(a) = \int_0^\infty g(p)\mathbb{1}_{\{p \geq V'(a)\}}dp$ is log-concave, by Prékopa (1973). $\square$

This result implies that the seller finds it increasingly hard to trade as he unwinds his position, and it grows harder to predict the next trade time. But this is even stronger than stated. For given the log-convexity result, not only do the mean waiting time $\tau(a)$ and variance $\xi(a)$ increase as the asset is sold off, but they *proportionately* increase — for instance, the hazard rate $-\tau'(a)/\tau(a)$ rises as a falls.

The proof of Theorem 7 in §A.8 establishes more than is claimed. In particular, it shows that the elasticity $|\mathcal{E}_\rho(\Phi(a))|$ falls in the position: The trade chance falls in
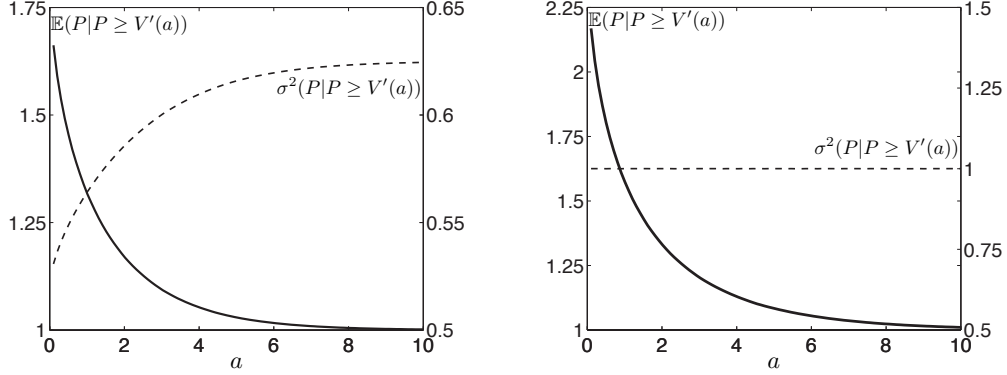
15

Figure 6: **Expected price and variance while unwinding.** The expected trade price increases and variance falls. At left, when $P \sim \Gamma(1.6, 1/1.6)$, so $\mathbb{E}(P) = 1$ and $s(P) = 0.625$. At right, when $P \sim \Gamma(1, 1)$ $\mathbb{E}(P) = 1$ and $s(P) = 1$, that remains unchanged. Both numerical dynamic programming simulations assume $X \sim \Gamma(0.5, 1)$.

the meeting rate ρ proportionally more at smaller positions. This monotone elasticity claim also holds for the interest rate and dividend. Namely, both $|\mathcal{E}_r(\Phi(a))| = \varphi(V'(a))(-V'_r(a))r$ and $|\mathcal{E}_k(\Phi(a))| = \varphi(V'(a))V'_k(a)k$ are decreasing in the position. That trading behavior is less responsive to search frictions and dividends at larger positions is consistent with the diminishing search optionality found in Lemma 2. In fact, all elasticities are less than unity at high search frictions — namely, large r or small ρ.

Let's now consider transactional liquidity. First, let's consider predictions about trade prices. Since the expected trade price $\mathbb{E}(P|P \geq V'(a))$ falls in the position a, by Theorem 1, the seller holds out for better prices while unwinding his position. But as seen in Figure 6, given the log-concave price density $g(p)$, the variance of traded prices $Var(P|P \geq V'(a))$ is nondecreasing in the position a (Heckman and Honoré, 1990). So as the seller sells his position, his terms of trade improve and grow more predictable, and the *expected mark-up* $\mathbb{E}[P - V'(a)|P \geq V'(a)]$ falls.[5]

Finally, we consider transactional liquidity from finance relating price and quantity. Our seller is analogous to the long-lived insider in Kyle (1985), whom we will assume, for definiteness, wishes to sell. His derives a trading rule that optimally trades off exploiting his informational edge and securing its fruits. In solving his dynamic arbitrage, he finds that the *market depth* — namely, the absolute slope of the demand curve — is optimally constant in time and price. Our seller also has a dynamically optimal selling schedule that trades off price and quantity at each instant, and we likewise can use this depth measure. But here we see the critical role of Kyle's informational asymmetry: *For his insider depresses the price by selling more today, whereas our seller throws away future*

---

[5]By Keynes (1930), the asset is more liquid with a smaller position: "One asset is more liquid than another one if it is more certainly convertible into money at short notice without loss."
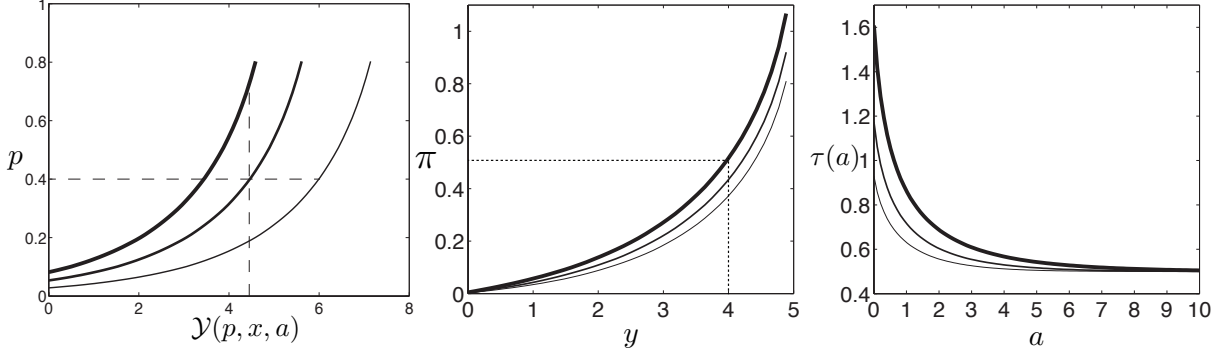
Figure 7: **Transactional Liquidity and Time to Trade**. When $k = 0$ and $a = 5$. At left is the elasticity $\eta(p, a)$ through the horizontal dashed line, and depth $\lambda(y, a)$ through the vertical one. At center, the purchase premium as a function of trade size. The right panel depicts the expected time to trade $\tau(a)$ as a function of a. In the center and right panels, frictions $\psi = r/\rho$ increase from thick to thin lines: $0.005, 0.1$ and 0.5.

options by selling more. Our seller only sells more when offered a higher price, a standard assumption of supply curves.[6] Depth is an inverse measure of the price impact of trades. In our setting, it is the slope of the residual inverse supply —- stealing Kyle's notation for our different trading context, we denote it by $1/\lambda(y, a) = -1/V''(a - y)$. Unlike Kyle's competitive model, depth is not constant: It increases in the position and falls in the trade size, by Theorem 4.

We can also address a property of liquid markets. For trades $y < a$, our seller charges a *purchase premium* $\pi(y, a) = V'(a - y) - V'(a)$ over his choke price, since his search optionality is proportionately more valuable with a smaller position. Analogous to a tightness measure, this premium is intuitively smaller in a more liquid market, since a trader has a smaller price impact.[7] We next explore measures of transactional liquidity.

Theorem 8 (*i*) *Supply* $\mathcal{Y}(p, x, a)$ *is nondecreasing in* a, r, *and nonincreasing in* ρ, k.
(*ii*) *The supply elasticity* $\eta(p, a)$ *is decreasing and convex in* a, *and vanishes as* $a \to \infty$.
Depth $1/\lambda(y, a)$ *is increasing in* a *and decreasing in* y. *It falls in* ρ *and* k, *and rises in* r.
(*iii*) *The purchase premium* $\pi(y, a)$ *is increasing in the trade size* $y < a$ *and decreasing in assets* a. *It falls in* r, *and rises in* ρ *and* k.

As the position a explodes, the optimal sales policy converges to a stationary rule — the seller avails himself fully of all limit offers with prices $p \geq k/r$, and otherwise abstains. Indeed, $V'(a) \to k/r$ as $a \to \infty$, by Lemma 2. But sales stochastically drift down as the seller's position unwinds. For the seller's own cap starts to bind more than

---

[6]A different dynamic tradeoff arises in each case: Kyle's seller optimally exploits his informational edge over time (Anderson and Smith, 2013), whereas ours optimally exploits his search optionality.

[7]In a two sided version of this paper that we have begun, this will analogize to a bid-ask spread.

17

the purchase caps, and he simultaneously grows more choosy due to value concavity — e.g. his choke price rises. A nearly stationary rule is once again optimal for small positions a, selling out for any price $p > V'(0+)$, and the purchase caps don't bind.

Likewise, as the asset position falls, the seller's optionality value figures more prominently in his optimization (Lemma 2). Accordingly, his purchase premium rises and his depth falls — the ask-price grows more responsive to the trades. This suggests that as time passes, liquidity worsens. Yet paradoxically, this supply elasticity rises (Figure 7). As the position falls to zero, the seller exploits the divisibility of the asset less, and his prices all converges to the sell-all price.

It might seem intuitive that liquidity worsens with greater search frictions. We find, quite the contrary, that *transactional liquidity improves with search frictions* — i.e., a higher interest rate or a lower arrival rate.[8] Theorem 8 asserts that the seller's chosen trade volume[9] and depth rises, and the purchase premium falls. On the other hand, more search frictions worsens one transactional liquidity measure — trade prices fall and grow more volatile, and the expected markup rises.[10] For V' falls, by Theorem 5.

The new dimension of liquidity afforded by this search framework, the waiting time between transactions, has an ambivalent response to increased search frictions. With lower interest rates, by Theorem 7, sales slow down, and the seller waits longer to trade.[11] But with a higher arrival rate ρ, sales accelerate, and so waiting times fall.

7   Search and Nash Bargaining

Trade opportunities only arrive periodically, and so are intuitively subject to negotiation. Nash bargaining over prices is commonly assumed in the monetary search literature. We argue below that we can easily modify our model to incorporate the Nash bargaining solution. We show that our results all naturally extend, but that the model is now richer, and affords further results about the negotiated trade price, quantity, and trade value.

For expositional purposes, consider the case of a land owner (the seller) liquidating his production stock in strawberries. Offers arrive randomly, when a buyer stops by

---

[8]E.g., trading volume rises at higher interest rates (Proposition 2 in Anderson and Smith (2013)).

[9] Trade volume does not fall too fast, for we show in §A.9 that the expected sales rate $\rho\mathbb{E}[\mathcal{Y}(P, X, a)]$ increases in the arrival rate ρ, when ρ is low.

[10]It might seem puzzling that the expected markup $\mathbb{E}[P - V'(a)|P \geq V'(a)]$ rises and yet the purchase premium falls. But the seller's first order condition $p - V'(a - y) = 0$ only holds for interior solutions. Absent purchase caps, it would be an identity, and the expected markup equals the premium $\pi(y, a)$.

[11]Our liquidation model sheds light on Alan Greenspan's comment: "Super slow interest rates can actually slow the process of liquidation, because the cost of carrying debt is so low" (Leonard and Coy, 2012). As argued in §4, the seller minimizes the opportunity cost $pa' - V(a')$ of holding a position a'.

driving his vehicle. Buyers vary in their willingness to pay w and in their carrying capacity x, since some drive bicycles, some small cars, some pickup trucks.

Assume bargaining weights $\delta \in [0,1]$ and $1 - \delta$ on the surplus of the seller and buyer, respectively. The seller's surplus of trading (over not trading) is $s(p, y, a) = py + \mathcal{V}(a-y) - \mathcal{V}(a)$, and the buyer's surplus is $(w-p)y$. The terms of trade dictated by the Nash solution entail a *negotiated price* $\mathcal{P}$ and *bargained supply* Υ functions obeying:

$$(\mathcal{P}(w,x,a), \Upsilon(w,x,a)) \equiv \underset{\{p, 0 \leq y \leq \min\{x,a\}\}}{\arg\max} s(p,y,a)^{\delta}((w-p)y)^{1-\delta} \tag{8}$$

We solve this maximization in stages. The FOC in p suffices by concavity in p. Given the reservation offer (w, x), the negotiated price is the weighted average of the two parties' reservation prices:

$$p = \delta w + (1-\delta)(\mathcal{V}(a) - \mathcal{V}(a-y)/y \tag{9}$$

The seller and buyer respectively secure fractions δ and $1 - \delta$ of the *total surplus* $\mathcal{S}(w,x,a)$. The bargained supply must maximize total surplus, namely, $\Upsilon(w,x,a) = \arg\max_{y \in [0, \min\{x,a\}]} \mathcal{S}(w,x,a)$, exactly as in our original model. We offer some insights:

Observe that *the Nash bargaining model is formally equivalent to the original model with a lower arrival rate $\rho\delta$ of offers (w,x) drawn from the density f.* For since the seller is risk neutral, we can imagine that he secures price w with chance δ and otherwise gets his reservation (zero surplus) price. We recover our original model with $\delta = 1$. The case $\delta = 0$ erases all trade surplus, and the seller holds assets for their dividend stream, i.e. $\mathcal{V}(a) = ak/r$. Hence, *greater buyer bargaining power is formally equivalent to higher search frictions.* The seller's value, marginal value and absolute second derivative are scaled lower with bargaining, since $\mathcal{V}(a|\rho,\delta) \equiv V(a|\delta\rho)$ and $\mathcal{V}'(a|\rho,\delta) \equiv V'(a|\delta\rho)$, and recalling Theorem 5. Given this logic, we now review how bargaining impacts our results.

1. We first observe that *the negotiated price and bargained supply rise in the cap.* The price is $p = \mathcal{P}(w,x,a)$ in (9) when evaluated at $y = \Upsilon(w,x,a)$. Since bargained supply $\Upsilon(w,x,a)$ rises in x, so too does $\mathcal{P}(w,x,a)$, by concavity of the value function.

For example, assume two buyers with the same reservation price w. If one drives a large truck, and the other rides a bike, then we predict that the truck driver buys more than and yet pays a steeper price: There is no volume discount! The reason owes to the option value of asset. For the seller's marginal benefit is constant at w, while the seller's marginal cost is rising in the quantity sold, by the concavity of the value function.

2. We next note that *greater bargaining power for buyers raises supply and lowers the negotiated price, the choke price, and the sell-all price:* The bargained supply $\Upsilon(w,x,a)$

is given by (4) but with meeting rate $\rho\delta$. By Theorem 8(i), it falls in the seller's bargaining power δ. Since the buyer secures a fraction $1 - \delta$ of total surplus $\mathcal{S}(w, x, a)$:

$$[w - \mathcal{P}(w, x, a)]\Upsilon(w, x, a) = (1 - \delta)\mathcal{S}(w, x, a) \tag{10}$$

Recall that $\mathcal{S}(w, x, a) = \max_{y \in [0, \min\{x, a\}]} \int_0^y (p - \mathcal{V}'(a - t))dt$ falls in δ by Theorem 5. In the corner solution when $\Upsilon(w, x, a) = \min\{x, a\}$, the price $\mathcal{P}(w, x, a)$ rises in δ. We now claim that this holds generally when $\Upsilon(w, x, a) < \min\{x, a\}$ and $w \equiv \mathcal{V}'(a - \Upsilon(w, x, a))$. For define the trade surplus $s(y, a) \equiv \mathcal{V}(a - y) + \mathcal{V}'(a - y)y - \mathcal{V}(a)$, and rewrite (10) as $\mathcal{P}(w, x, a) = w - (1 - \delta)s(\Upsilon(w, x, a), a)/\Upsilon(w, x, a)$. Appendix Lemma A.1 verifies that $s(y, a)/y$ rises in y. Thus, $s(\Upsilon(w, x, a), a)/\Upsilon(w, x, a)$ falls in δ, since $\Upsilon(w, x, a)$ falls in δ. Finally, the two threshold prices fall as $\mathcal{V}'(a) < V'(a)$.

The logic of this point implies that with greater search frictions, not only does the bargained supply increase (as is true without bargaining), but the negotiated price falls.

3. Next, *the bargained supply rises in the position, and the negotiated price falls.* Supply rises just as in (4). Substitute the optimal supply $y = \Upsilon(w, x, a)$ into (8). This reduces to (∗): $\Upsilon(w, x, a)(p - c(\Upsilon(w, x, a), a))^\delta (w - p)^{1-\delta}$, where the secant slope of $\mathcal{V}$ is

$$c(y, a) = [\mathcal{V}(a) - \mathcal{V}(a - y)]/y = \int_0^1 \mathcal{V}'(a - (1 - t)y)dt \tag{11}$$

First, by Topkis (1978), the price $\mathcal{P}(w, x, a)$ rises in a since (∗) is log-supermodular in $(p, -a)$ — as its middle factor $(p - c(\Upsilon(w, x, a), a))^\delta$ is log-supermodular in $(p, -a)$. For the slope of supply $\Upsilon(w, x, a)$ in a is at most one, as in (4). So substituting $y = \Upsilon(w, x, a)$ in (11), the argument $a - (1 - t)y$ rises in a, i.e. $c(\Upsilon(w, x, a), a)$ falls in a, as $\mathcal{V}$ is concave.

4. *The trade value is increasing and concave in the position a until the cap binds, and then decreasing and convex.* Without bargaining, the trade value plot mirrors the supply (4), since the price is fixed — it is piecewise linear in the position a, rising with slope w until $V'(a - \min\{x, a\}) = w$, and then is constant. With bargaining, the trade value $\mathcal{P}(w, x, a)\Upsilon(w, x, a)$ initially vanishes, then is increasing and *strictly* concave in a until $\mathcal{V}'(a - \min\{x, a\}) = w$, and then decreasing and strictly convex.[12] For supply is fixed at x, but the price is decreasing and strictly convex in the asset position a.

5. *Bargaining lowers the trade value, except for low reservation prices and positions.* For reservation values w above the sell-all price $V'(0+) > \mathcal{V}'(0+)$, supply is unchanged and the price is lower, and so the trade value lower. Next, consider lower w. The

---

[12]Indeed, by (9), the trade value is $\delta w \Upsilon(w, x, a) + (1 - \delta)(\mathcal{V}(a) - \mathcal{V}(a - \Upsilon(w, x, a)))$, and $a - \Upsilon(w, x, a)$ is constant in a when the purchase cap does not bind, by (4). Finally, $\Upsilon(w, x, a)$ is piecewise linear, and $\mathcal{V}'(a)$ falls and is strictly convex.
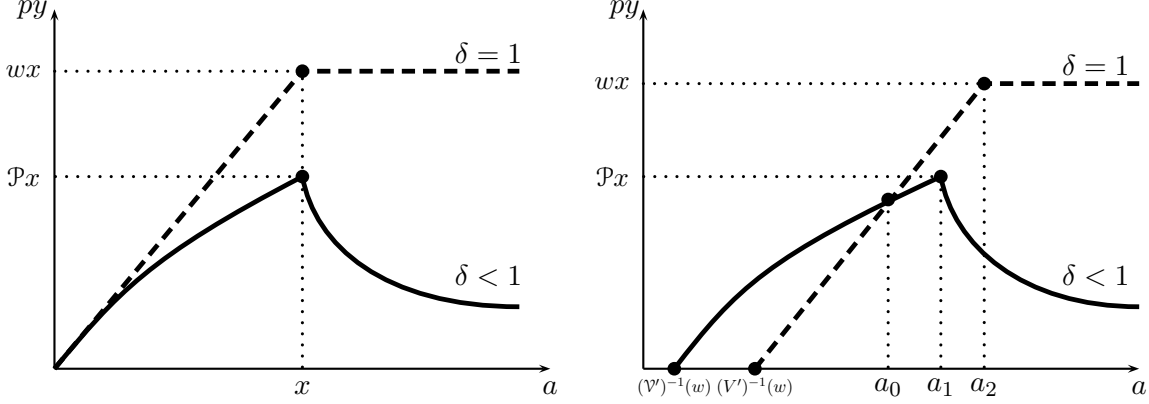
Figure 8: **Bargaining and the Trade Value**. We plot the trade value with and without bargaining (thick and dashed lines). At left, $w\mathcal{Y}(w,x,a) > \mathcal{P}(w,x,a)\Upsilon(w,x,a)$ when $w > V'(0+)$. At right, when $w \leq V'(0+)$, the trade value rises for low positions $a \leq a_0$. At positions a_1 and a_2, the purchase cap binds with and without bargaining.

bargained supply $\Upsilon(w,x,a)$ has unit slope in a, and surplus $\mathcal{S}(w,x,a)$ rises in a. So from (10), the trade value $\mathcal{P}(w,x,a)\Upsilon(w,x,a)$ has slope at most w in a. But the slope in a of the trade value $w\mathcal{Y}(w,x,a)$ without bargaining is w, recalling Theorem 1. Since the maximum trade revenue $\mathcal{P}(w,x,a)\Upsilon(w,x,a)$ lies below its no-bargaining counterpart wx, and falls after peaking, the two trade values cross (Figure 8).

6. *Greater bargaining power for buyers lowers the mean and variance of waiting times.* As in Theorem 7, this follows because the chance of a desirable trade $(1 - F(\mathcal{V}'(a), \infty))$ is now higher — because $\mathcal{V}'$ rises in ρ and thus in δ, by Theorem 5.

7. *Greater bargaining power for buyers raises depth and lowers the purchase premium.* For the FOC $w \equiv \mathcal{V}'(a - y)$ implies the *inverse uncapped supply curve* (9):

$$p(y, a) = \delta\mathcal{V}'(a - y) + (1 - \delta) \int_0^y \mathcal{V}'(a - t)dt/y \tag{12}$$

Firstly, depth is the inverse slope $\Lambda(y, a) = (\partial p(y, a)/\partial y)^{-1}$, and the uncapped supply has slope $p_1(y, a) = -\delta\mathcal{V}''(a - y) + (1 - \delta)c_1(y, a)$, where $c_1(y, a) = -y^{-2} \int_0^y \int_{a-y}^{a-t} \mathcal{V}''(u)dudt$, recalling (11). Next, using (12), rewrite the purchase premium $\Pi(y, a) = p(y, a) - \mathcal{V}'(a)$ as

$$\Pi(y, a)y = \delta \int_0^y (\mathcal{V}'(a - y) - \mathcal{V}'(a - t))dt + \int_0^y (\mathcal{V}'(a - t) - \mathcal{V}'(a))dt$$

By our equivalence result, it suffices that $\mathcal{V}''$ fall in ρ and thus in δ (true by Theorem 5).

8. *The qualitative behavior of market depth, the purchase premium, and the sales elasticity claimed in Theorem 8 still hold with bargaining,* as verified in Appendix A.10.

21

8 Conclusions

The large search theory literature in economics has always assumed that individual optimizations either involve indivisible units, or only last a single period (such as before access to a fixed outside option, such as a market). This paper develops a tractable approach to fully dynamic search for perfectly divisible units. Our theory proposes a dynamic search model founded on Bellman value functions, and not just Bellman values. We uncover two insights about the value and the marginal value of assets: First, the value is a strictly concave function of the position. This diminishing returns to optionality has important implications. For instance, barring a high price, the seller is often only willing to partially unwind his position — i.e., "search at the margin". This also means that the seller is choosier with a smaller position, since the search optionality matters more.

Our second novel finding is that the marginal value of assets is convex. This has many implications for liquidity — e.g., with a smaller position, the seller imposes a higher purchase premium over his choke price. Also, our new measure of transactional liquidity — the waiting time between trades — is log-convex in the asset position. Inasmuch as the counterparty arrival process in finance is better approximated by periodic arrivals than a constant flow, this search theory story of the liquidation process is a useful alternative framework for understanding liquidity that is amenable to variable bargaining power.

The single agent search literature has long been essentially closed. All research now builds on a well-established optimization toolset in equilibrium analysis. We add a tool to the search theorists' toolbox. For our model explores a purposefully simple optimization exercise, and our underlying framework is adaptable to a variety of contexts. We hope that it can be a key ingredient in a wealth of equilibrium analyses in which the buyers' behavior is derived and not exogenously specified. Our paper should allow, e.g., multiple periods of search before markets open in money papers.

It is tempting to ask why our asset value function shares the same first three derivative properties as assumed of the utility for money. But money is one such asset that provides an option value for future trades. If those opportunities are bounded in size, and have a vanishing tail, then the *induced utility for money* is concave with a convex marginal utility. This suggests that consumer theory under risk aversion may more broadly apply to firms (hereby thinking of the motives of firms like Apple to hold so much cash).

We have analyzed a "pure" problem with a homogeneous asset, having a scalar summary. With many asset classes, one should apply our theory separately for each class.

We are currently extending the analysis to a middleman managing his inventory, who both buys from periodically arriving sellers, and sells to periodically arriving buyers.

A Appendix: Omitted Proofs

A.1 The Surplus Recursion: Proof of Lemma 1

We first show that $\mathcal{T}$ maps $\mathcal{B} \to \mathcal{B}$ with bound $\rho \mathbb{E}[PX]/r$. Relaxing the asset constraint in (1), if we assume $|W| \leq \rho \mathbb{E}[PX]/r$, then we see that $\mathcal{T}$ preserves the upper bound:

$$(\mathcal{T}W)(a) \leq \rho \mathbb{E}(\max_{y \in [0,X]}[Py + \rho \mathbb{E}[PX]/r])/(r+\rho) \leq \rho \mathbb{E}[PX]/r$$

Next, as the maximum of a continuous function $(p-k/r)y + W(a-y)$ yields a continuous function in a, and $f(p,x) \in \mathcal{B}$, we have $\mathcal{T}: \mathcal{B} \to \mathcal{B}$. We next verify the Blackwell sufficient conditions for a contraction. By inspection, $\mathcal{T}$ is monotone. Likewise, $\mathcal{T}(W+b)(a) \leq (\mathcal{T}W)(a) + \beta b$, where $\beta = \rho/(r+\rho) < 1$. Since $\mathcal{B}$ is a complete metric space with the sup norm, by the Contraction Mapping Theorem, $\mathcal{T}W = W \in \mathcal{B}$ is unique. $\qquad\square$

A.2 Monotonicity of Recursion Operator: Proof of Theorem 1

Since $\mathcal{T}$ is a contraction, and monotonicity is a closed property in $\mathcal{B}$ with the sup norm, it suffices to show that is preserved by $\mathcal{T}$ (Corollary 3.2.1 in Lucas, Stokey, and Prescott (1989)). From (1), since the choice set and the objective function increase in a, then $(\mathcal{T}W)(a) \leq (\mathcal{T}W)(a')$, so the fixed point increases in a. Second, $W(a)$ is strictly increasing, for one sales strategy available at $b > a$ is to act as if one's position is a, and for unexploited offers $\overline{F}(0,a) > 0$ sell at any positive price, which yields extra payoff at least $\mathbb{E}[P\min\{b-a, X-\min\{X,a\}\}] > 0$. So $V(a) = W(a) + ak/r$ is strictly increasing.

A.3 Value Function Differentiability: Proof of Theorem 2

We show that there exists a unique bounded continuous function $U = \mathcal{S}U$, recalling (3). We also show that any such fixed point U is the derivative of $V(a) = W(a) + ak/r$, where $W = \mathcal{T}W$, in other words, $U = V'$. We attack these tasks in reverse order.

STEP 1. We know from Lemma 1 that V satisfies the Bellman equation:

$$V(a) = \frac{ak}{r+\rho} + \frac{\rho}{r+\rho}\mathbb{E}\left(\max_{y \in [0,\min\{X,a\}]}[Py + V(a-y)]\right) \tag{13}$$

If V is continuously differentiable on $[0,\infty)$, then Corollary 1 is valid (as its proof

only exploits concavity of V). So Corollary 5 in Milgrom and Segal (2002) applies:

$$\frac{\partial}{\partial a}\mathbb{E}\left(\max_{y\in[0,\min\{X,a\}]}[Py+V(a-y)]\right) = \mathbb{E}[V'(a-\mathcal{Y}(P,X,a)) + \lambda_2(\mathcal{Y}(P,X,a))]$$

Now, $\lambda_2(\mathcal{Y}(p,x,a)) = 0$ for interior solutions or if $\mathcal{Y}(p,x,a) = 0$. So, $V'(a-\mathcal{Y}(p,x,a)) + \lambda_2(\mathcal{Y}(p,x,a)) = \max\{V'(a),p\}$ when $p \leq V'(a-\min\{x,a\})$. Otherwise, the multiplier $\lambda_2(\mathcal{Y}(p,x,a))$ vanishes only when $x \leq a$, so that $V'(a-\mathcal{Y}(p,x,a)) + \lambda_2(\mathcal{Y}(p,x,a)) = \min\{p,V'(a-\min\{x,a\})(1+\chi_{[0,a]}(x))\}$. Hence, $\mathbb{E}[V'(a-\mathcal{Y}(p,x,a)) + \lambda_2(\mathcal{Y}(p,x,a))] = \mathbb{E}[\max\{V'(a),\min\{p,V'(a-\min\{x,a\})(1+\chi_{[0,a]}(x))\}\}]$. Add $k/(r+\rho)$ to get (3). So altogether, if (13) has a differentiable solution, then its derivative satisfies (3).

STEP 2. We show that $\mathcal{S}$ is a contracting operator, and so has a unique bounded and continuous fixed point $U = \mathcal{S}U$. Assume $0 \leq U(a) \leq k/r + \rho\mathbb{E}[P]/r$ and that U is continuous. Since $0 \leq (\mathcal{S}U)(a) \leq k/r + \rho\mathbb{E}[P]/r$ by (3), the bound $\rho\mathbb{E}[P]/r$ is preserved.

To see that $\mathcal{S}$ preserves continuity, let $a \geq 0$, and consider any sequence $a_n \to a$. Let

$$\mu_n(p,x) =: \max\{U(a_n), \min\{p, U(a_n - \min\{x,a_n\})(1+\chi_{[0,a_n]}(x))\}$$

Since $\mu_n(p,x) \leq U(a_n) + p \leq \rho\mathbb{E}[P]/r + p < \infty$, the Dominated Convergence Theorem implies $\lim_{n\to\infty}\mathbb{E}[\mu_n(P,X)] = \mathbb{E}[\lim_{n\to\infty}\mu_n(P,X)]$. Then $\lim_{a_n\to\infty}\mathcal{S}U(a_n) = \mathcal{S}U(a)$.

Next, $\mathcal{S}$ obeys the Blackwell sufficient conditions for a contraction: $\mathcal{S}$ is monotone, and $\mathcal{S}(U+b)(a)\leq(\mathcal{S}U)(a)+\beta b$, where $\beta=\rho/(r+\rho)<1$. Namely, $\mathcal{S}U = U \in \mathcal{B}$ is unique.

For the last statement we use $V'(0+) < \infty$ and the concavity of V to establish $0 < \overline{F}(V'(0+),0) \leq \overline{F}(V'(a),0)$. In words, there is a strictly positive mass of prices such that $\max\{V'(a),p\} > V'(a)$. We use this in (3) to show that $(\mathcal{S}V')(a) > k/(r+\rho)+\rho V'(a)/(r+\rho)$. Finally, if $V'(a) \geq k/r$, then $(\mathcal{S}V')(a) > k/r$. Hence, $SV' = V' > k/r.\square$

A.4   Strict Value Concavity: Proof of Theorem 3

Since V is concave (Theorem 1), it suffices that V be linear on no interval. Assume not.

Assume first that V is linear on $[0,a_2]$, say $V(a) = ca$ for $c < \infty$ and $a_2 > 0$. Then the optimal policy in (1) for position $a \in [0,a_2]$ is to sell $y = \min\{x,a\}$ if $p \geq c$, and $y = 0$ otherwise. Define the *survivor* $\overline{F}(p,x) = \int_x^\infty \int_p^\infty f(s,t)dsdt$ and the cdf $F(p,x) = \int_0^x \int_0^p f(s,t)dsdt$. As $V(a)=W(a)+ak/r$, by (1), *identically* in interval $[0,a_2]$:

$$c(r+\rho)a \equiv a\left(k+\rho c+\rho\int_c^\infty \overline{F}(p,0)dp\right) - \rho\int_0^a\int_c^\infty[1-\overline{F}(0,x)-F(p,x)]dpdx \quad (14)$$

Thus, the derivative $\rho\int_c^\infty\int_p^\infty f(s,a)dsdp$ of the second term on the right side of (14) is

24

constant. This forces $c = \infty$, since its derivative is constant at 0. A contradiction.

Next assume V is linear on $[a_1, a_2]$, with $a_1 > 0$. By (3), for any position $a \in [a_1, a_2]$,

$$c = \frac{k}{r+\rho} + \frac{\rho}{r+\rho} \int_0^\infty \int_0^\infty \max\left\{c, \min\left\{p, V'(a - \min\{x,a\})(1 + \chi_{[0,a]}(x))\right\}\right\} dF(p, x) \quad (15)$$

Let $J(a)$ be the right side of (15). Write the max term in $J(a_i)$ as $\max(c, \min(p, \gamma_i \zeta_i))$ for $i = 1, 2$. Since $a_2 > a_1$, we have $\gamma_2 \leq \gamma_1$ by concavity of V, and $\zeta_2 \leq \zeta_1$. Thus,

$$\max(c, \min(p, \gamma_2 \zeta_2)) - \max(c, \min(P, \gamma_1 \zeta_1)) \leq \max(c, \min(P, \gamma_1 \zeta_2)) - \max(c, \min(p, \gamma_1 \zeta_1))$$

The only non-zero terms on the right side arise for $\zeta_1 > \zeta_2$ — i.e. $a_1 < x < a_2$ — and $p > V'(0+)$. Taking integrals, $V'(a_2) - V'(a_1) = \rho \int_{a_1}^{a_2} \int_{V'(0+)}^\infty [V'(0+) - p] dF(p, x) < 0$. $\quad \square$

A.5 Strictly Convexity of Marginal Value: Proof of Theorem 4

First, we write the continuation marginal value in (3) as

$$V'(a) + \sigma'(a) = \mathbb{E}[\max\{P, V'(a)\}] - \int_0^a \int_0^\infty \max\{p - V'(a - x), 0\} dF \quad (16)$$

The first term on the right side reflects the indivisible asset logic. The second subtracts how much the price exceeds the marginal value for offers with a binding purchase cap. Rewrite (16) using the uncapped supply curve $Y(p, a) = a - (V')^{-1}(\min\{p, V'(0+)\})$ derived in (4). Use $\int_0^u (u - x) dF(x) = \int_0^u F(x) dx$ and $\int_b^\infty (x - b) d\overline{F}(x) = -\int_b^\infty \overline{F}(x) dx$ to integrate by parts, and change the order of integration. We arrive at the expression:

$$V'(a) + \sigma'(a) = \mathbb{E}[P] + \int_0^{V'(a)} F(p, \infty) dp - \int_0^\infty \int_0^{Y(p,a)} \int_p^\infty f(s, t) ds \, dt \, dp \quad (17)$$

Next, write the right side of the V' operator in (3) as $(\mathcal{S}V')(a) = k/(r + \rho) + \beta(V'(a) + \sigma'(a))$ where $\beta = \rho/(r + \rho)$. We now reformulate this expression using (17):

$$(\mathcal{S}V')(a) = \frac{k}{r+\rho} + \beta \left(\mathbb{E}[P] + \int_0^{V'(a)} F(p, \infty) dp - \int_0^\infty \int_0^{Y(p,a)} \int_p^\infty f(s, t) ds \, dt \, dp \right) \quad (18)$$

Assume that $V'(a)$ is convex in a. Since $\int_0^u F(p, \infty)$ is increasing and convex in u, by Theorem 5.1 in Rockafellar (1970), $\int_0^{V'(a)} F(p, \infty) dp$ is convex in a. Since $Y_2(p, a) \equiv 1$, the second integral in (18) is also convex in a since its derivative in a, namely $-\int_0^\infty \int_p^\infty f(s, Y(p, a)) ds dp$, is increasing in a. In summary, $\mathcal{S}$ preserves convexity, a closed property under the sup norm. So the fixed point $\mathcal{S}V' = V'$ is convex in a.

To prove strict convexity, from the second conclusion of Corollary 3.2.1 in Lucas,

25

Stokey, and Prescott (1989), it suffices that the image of a convex function V' is strictly convex. This follows because $\int_0^\infty \int_p^\infty f(s, Y(p,a))dsdp$ strictly falls in a since $f(s,y)$ must eventually strictly fall in y, as $y \to \infty$, or the expected price would be infinite.

We now show that V' is differentiable in a, paralleling the proof of Lemma 2 in §A.3.

STEP 1. Assume V' is continuously differentiable on $[0, \infty)$. Differentiating (3):

$$V''(a) = \beta \left(\int_{V'(0+)}^\infty (V'(0+) - p)f(p,a)dp + F(V'(a), \infty)V''(a) + \int_0^a \int_{V'(a-x)}^\infty V''(a-x)dF \right) \text{(19)}$$

Any solution V'' of (19) is the derivative of V', if it is differentiable. As in §A.3, it suffices that for any given V', that exists a continuous function V'' satisfying (19).

STEP 2. We show that the right side of (19) defines a contraction mapping $\mathcal{H}$, and so has a unique bounded and continuous fixed point V'', with $-(\rho/r)\bar{p}\bar{h} \leq V''(a) \leq 0$ for all $a > 0$. Take $V'' \in \mathcal{B}$ with $-(\rho/r)\bar{p}\bar{h} \leq V''(a) \leq 0$. Since $V'(0+) > 0$, the first integral in (19) is negative and exceeds $-\bar{p}\bar{h}$. The last two terms in $(\mathcal{H}V'')(a)$ are negative with sum at least $-(\rho/r)\bar{p}\bar{h}$, by the assumed bound on V''. So $0 > (\mathcal{H}V'')(a) \geq -\beta(\bar{p}\bar{h} + (\rho/r)\bar{p}\bar{h}) = -(\rho/r)\bar{p}\bar{h}$. Since $V'(a)$ and $f(p,a)$ are both continuous in a, $\mathcal{H}:\mathcal{B}\to\mathcal{B}$.

Next, $\mathcal{H}$ obeys the Blackwell sufficient conditions for a contraction: First, $\mathcal{H}$ is monotone in V''. Next, since $V'(a) \leq V'(a-x)$, we have $F(V'(a), \infty) + \int_0^a \int_{V'(a-x)}^\infty dF \leq 1$. Then $(\mathcal{H}V'' + b)(a) \leq (\mathcal{H}V'')(a) + \beta b$. So $\mathcal{H}$ has a unique fixed point $\mathcal{H}V'' = V'' \in \mathcal{B}$. □

A.6 Supply Elasticity: Proof of Corollary 3

Observe that $p_1(a)Y_1(p_1(a), a) - Y(p_1(a), a) \geq 0$, with equality when $p_1(a) < V'(0+)$, as in Figure 2. So, if $\mu V'(0+) \geq a$, then $p_1(a) = V'(0+)$, since $Y(V'(0+), a) = a$ from (4). Otherwise, if $\mu V'(0+) < a$, then $p_1(a)Y_1(p_1(a), a) - Y(p_1(a), a) = 0$, so that $p_1(a) < V'(0+)$. Summarizing, $\eta(p,a) \gtrless 1$ iff $pY_1(p,a) \gtrless Y(p,a)$ iff $p \lessgtr p_1(a)$. □

A.7 Increasing Search Frictions: Proof of Theorem 5

Since $\mathcal{T}$ and $\mathcal{S}$ are monotone operators, any parametric change that increases the operator increases its fixed point. In particular, since $\mathcal{S}$ falls in r and rises in k, so do the marginal value V' and the value V. Rephrasing monotonicity of V', we see that V is strictly submodular in (a,r) and strictly supermodular in (a,k). Equivalently, since $\mathcal{T}$ rises in ρ, so does the option value W and thus the value V.

We show that V' increases in either parameter $\theta = -r, \rho, k$, where V' solves $V'(a) =$

26

$\mathcal{S}(V', \theta)(a)$, recalling (3). By the Lemma in Albrecht, Holmlund, and Lang (1991), V' is differentiable in a parameter θ, and the continuous derivative is the unique solution to $V'_\theta(a) = \mathcal{S}_\theta(V', \theta)(a) + \mathcal{S}_{V'}(V'_\theta, V', \theta)(a)$, whose right side is a contraction map in V'_θ:

$$(\mathcal{Q}V'_\rho)(a) = (-k + r(V'(a) + \sigma'(a)))/(r + \rho)^2 + \mathcal{S}_{V'}(V'_\rho, V', \rho)(a) \qquad (20)$$

$$(\mathcal{R}V'_{-r})(a) = (k + \rho(V'(a) + \sigma'(a)))/(r + \rho)^2 + \mathcal{S}_{V'}(V'_{-r}, V', -r)(a) \qquad (21)$$

$$(\mathcal{K}V'_k)(a) = 1/(r + \rho) + \mathcal{S}_{V'}(V'_k, V', k)(a) \qquad (22)$$

WLOG, let $\theta = \rho$, and focus on the $\mathcal{Q}$ recursion. If $V'_\rho \geq 0$ then $\mathcal{S}_{V'}(V'_\rho, V', \rho)(a) \geq 0$, and so the second term in (20) is nonnegative. Since $\sigma'(a) > 0$ by Corollary 2, we have $(r + \rho)^2(\mathcal{Q}V'_\rho)(a) > -k + rV'(a)$. Hence, $(\mathcal{Q}V'_\rho)(a) > 0$ as $V'(a) > k/r$ by Lemma 2. In light of the second conclusion of Corollary 3.2.1 in Lucas, Stokey, and Prescott (1989), the fixed point obeys $\mathcal{Q}V'_\rho = V'_\rho > 0$, and thus V is strictly supermodular in (a, ρ).

Next, we argue recursively from (20)–(22) that $V'_\theta(a)$ falls in a for $\theta = \{-r, \rho, k\}$. By the strict concavity of V, it suffices to show that $\mathcal{S}_{V'}(V'_\theta, V', \theta)(a)$ falls in a. Differentiate (16) in θ, and change the order of integration to integrate with respect to $z = a - x$:

$$\mathcal{S}_{V'}(\theta, V'; V'_\theta)(a) = \tfrac{\rho}{r+\rho}\left(V'_\theta(a)F(V'(a), \infty) + \int_0^\infty \int_{a-\mathcal{Y}(p,a,a)}^\infty V'_\theta(z)f(p, a - z)dzdp\right) \qquad (23)$$

The first term in (23) falls in a by the concavity of V, and the second since, by Theorem 1, $a - \mathcal{Y}(p, a, a) = \min\{a, (V')^{-1}(\min\{p, V'(0+)\})\}$ increases in a, and the density $f(p, a-z)$ falls in a. As a result, $SV'_\theta = V'_\theta$ falls in a for $\theta = \{-r, \rho, k\}$. $\qquad \square$

A.8   Waiting Times: Proof of Theorem 7 Finished

Define the hazard rate $\varphi(p) = F_1(p, \infty)/\overline{F}(p, 0)$. As V' is continuously differentiable in ρ (see §A.7), the elasticity $\mathcal{E}_\rho(\Phi(a)) = -\varphi(V'(a))V'_\rho(a)\rho$ is well-defined. Now, $V(a)$ is concave by Theorem 3 and $V'(a)$ is submodular in (a, ρ) by Theorem 5. Also, $\varphi(p)$ is increasing since $g(p)$ is log concave, and thus so too is $\overline{F}(p, 0)$. So the absolute elasticity $|\mathcal{E}_\rho(\Phi(a))|$ falls in a. So $|\mathcal{E}_\rho(\Phi(a))| \leq |\mathcal{E}_\rho(\Phi(0+))| = \varphi(V'(0+))V'_\rho(0+)\rho$. From (20):

$$(r + \rho)V'_\rho(0+) = \frac{-k}{r + \rho} + \frac{r}{r + \rho}\mathbb{E}(\max\{P, V'(0+)\}) + \rho V'_\rho(0+)F(V'(0+), \infty) \qquad (24)$$

Now, (3) implies $\mathbb{E}(\max\{P, V'(0+)\}) = ((r + \rho)V'(0+) - k)/\rho$. Substituting into (24):

$$\rho V'_\rho(0+) = \frac{r(V'(0+) - k/r)}{r + \rho\overline{F}(V'(0+), 0)} < V'(0+) - k/r \qquad (25)$$

27

Since $rV'(0+) = k + \rho\mathbb{E}[\max\{P - V'(0+), 0\}]$, recalling our parallel wage search problem, the right side of (25) is less than $\rho\mathbb{E}[P]/r$. Altogether, $0 \leq |\mathcal{E}_\rho(\Phi(a))| \leq \varphi(V'(0+))\rho\mathbb{E}[P]/r$. So, $|\mathcal{E}_\rho(\Phi(a))| \to 0$ as $\rho \to 0$ (since $V'(0+) \to k/r$). By the sandwich inequality, $|\mathcal{E}_\rho(\Phi(a))| \leq 1$ if ρ is low enough, say $\rho \leq \bar{\rho}$. Finally, $\mathcal{E}_\rho(\xi(a)) = 2\mathcal{E}_\rho(\tau(a)) = 2(-1 + |\mathcal{E}_\rho(\Phi(a))|) \leq 0$ if ρ is low enough. $\qquad\square$

A.9 Transactional Liquidity Measures: Proof of Theorem 8

Write the seller's objective function (1) as $py + V(a - y)$. This is supermodular in (y, θ), for $\theta = r, -\rho, -k$ by Theorem 5, $\mathcal{Y}(p, x, a)$ rises in θ (Theorem 6.1 in Topkis (1978)). By the same logic, $py + V(a - y) - \chi_{[0,a]}(y)$ is supermodular in (y, a), so $\mathcal{Y}(p, x, a)$ rises in a.

Since $V'(a)$ is decreasing and convex by Theorem 4, $\pi(y, a)$ is increasing in y, and decreasing in a. Likewise, $\pi(y, a)$ falls in $\theta = r, -\rho, -k$, since $V'(a)$ is supermodular in each pair (a, θ), by Theorem 5.

Supply is constant in p and equal to $\min\{x, a\}$ if $p \geq V'(0+)$ or constant in p and zero if $p \leq V'(a)$. Otherwise, the supply is $Y(p, a) = a - (V')^{-1}(p)$ and thus $\eta(p, a) = Y_1(p, a)p/Y(p, a)$. Since the supply is additively separable in (a, p), its slope is constant in a. Then $\eta(p, a)$ is decreasing and convex in a, since $p/Y(p, a) = p/(a - (V')^{-1}(p))$ is decreasing and convex in a. Clearly as $a \to \infty$, $\eta(p, a) \to 0$.

Finally, $1/\lambda(y, a) = -1/V''(a - y)$. This is decreasing in a and increasing in y by Theorem 4. It rises in θ since $V''(a)$ rises in θ by Theorem 5. $\qquad\square$

We finally prove a claim in footnote 9 that the sales rate $\rho\mathbb{E}[\mathcal{Y}(P, X, a)]$ rises in ρ, for low ρ. By Theorem 4, the supply $Y(p, a)$ is differentiable in the domain $\mathcal{D}(x, a) = \{p | V'(a) \leq p \leq V'(a - \min\{x, a\})\}$. The sales rate derivative is

$$\rho\partial\mathbb{E}[Y(p, a)]/\partial\rho + \mathbb{E}[\mathcal{Y}(P, X, a)] = \rho\left(\mathbb{E}[Y_\rho(P, a)\mathbb{1}_{P \in \mathcal{D}(X,a)}] + \mathbb{E}[\mathcal{Y}(P, X, a)]/\rho\right) \quad (26)$$

As $\rho \to 0$, we have $V'(a) \to k/r$ by Lemma 2, whence the domain $\mathcal{D}(X, a)$ collapses to a point. To wit, $\mathbb{E}[Y_\rho(P, a)\mathbb{1}_{P \in \mathcal{D}(X,a)}] \to 0$. The term $\mathbb{E}[\mathcal{Y}(P, X, a)]/\rho$ falls from ∞ to 0 as ρ increases. So for small ρ, the right side of (26) is positive, i.e. $\partial(\rho\mathbb{E}[\mathcal{Y}(P, X, a)])/\partial\rho \geq 0$.

A.10 Liquidity with Nash Bargaining: Proofs for §7

Recalling (12), $\mathcal{P}(w, x, a) \equiv p(\Upsilon(w, a), a)$, the bargained supply elasticity $\mathcal{H}(\mathcal{P}(w, x, a), a)$ solves:

$$\frac{1}{\mathcal{H}(\mathcal{P}(w, x, a), a)} = \frac{1}{\mathcal{P}(w, x, a)}\left(\frac{\delta w}{\mathcal{E}_w(\Upsilon(w, a))} + (1 - \delta)(w - c(\Upsilon(w, a), a))\right) \quad (27)$$

We have shown (in §7, point 4) that $c(\Upsilon(w,a),a)$ and $\mathcal{P}(w,x,a)$ fall in a, and Theorem 8 proves that the elasticity $\mathcal{E}_w(\Upsilon(w,x,a))$ falls in a. The right side of (27) rises in a. So the bargained supply elasticity $\mathcal{H}(\mathcal{P}(w,x,a),a)$ falls in the position a.

Next, since the two other transactional liquidity measures — depth and purchase premium — are expressed in terms of the inverse uncapped supply (12), it suffices to understand the marginal value $\mathcal{V}'$ and secant slope $c(y,a)$.

Lemma A.1 (Secant) *The secant slope $c(y,a)$ is increasing in y, ρ, k, and δ, falling in a and r, supermodular in $(y,-a)$, $(y,-r)$, (y,ρ), (y,δ), and (y,k), and convex in y.*

Proof: By the earlier equivalence result for the bargaining model, all monotonicity claims in Theorems 4–5 are inherited by $\mathcal{V}'$, and thus by $\int_0^y \mathcal{V}'(a-t)dt/y$. By the same logic, supermodularity claims about $c(y,a)$ follow from monotonicity of $c_1(y,a) = -y^{-2}\int_0^y \int_{a-y}^{a-t} \mathcal{V}''(u)dudt$. For the convexity in y, change the order of integration, and change variables, to get $c_1(y,a) = -\int_0^1 \mathcal{V}''(a-zy)zdz$. This rises in y by Theorem 4. $\square$

References

ALBRECHT, J. W., B. HOLMLUND, AND H. LANG (1991): "Comparative Statics in Dynamic Programming Models with an Application to Job Search," *Journal of Economic Dynamics and Control*, 15(4), 755–769.

ALIPRANTIS, C., G. CAMERA, AND F. RUSCITTI (2009): "Monetary equilibrium and the differentiability of the value function," *Journal of Economic Dynamics and Control*, 33(2), 454 – 462.

ANDERSON, A., AND L. SMITH (2013): "Dynamic Deception," *American Economic Review*, 103(7), 2811–47.

BENVENISTE, L. M., AND J. A. SCHEINKMAN (1979): "On the differentiability of the value function in dynamic models of economics," *Econometrica*, pp. 727–732.

GREEN, E. J., AND R. ZHOU (1998): "A rudimentary random-matching model with divisible money and prices," *Journal of Economic theory*, 81(2), 252–271.

HECKMAN, J. J., AND B. E. HONORÉ (1990): "The Empirical Content of the Roy Model," *Econometrica*, 58(5), pp. 1121–1149.

KEYNES, J. M. (1930): *Treatise on money: Pure theory of money Vol. I.* Macmillan and Co., London.

KIYOTAKI, N., AND R. WRIGHT (1989): "On money as a medium of exchange," *The Journal of Political Economy*, pp. 927–954.

KRAINER, J., AND S. F. LEROY (2002): "Equilibrium valuation of illiquid assets," *Economic Theory*, 19(2), 223–242.

KYLE, A. S. (1985): "Continuous Auctions and Insider Trading," *Econometrica*, 53(6), pp. 1315–1335.

LAGOS, R., AND G. ROCHETEAU (2009): "Liquidity in Asset Markets With Search Frictions," *Econometrica*, 77(2), 403–426.

LAGOS, R., G. ROCHETEAU, AND P.-O. WEILL (2011): "Crises and liquidity in over-the-counter markets," *Journal of Economic Theory*, 146(6), 2169 – 2205.

LAGOS, R., G. ROCHETEAU, AND R. WRIGHT (2015): "Liquidity: A New Monetarist Perspective," *Journal of Economic Literature, forthcoming.*

LAGOS, R., AND R. WRIGHT (2005): "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy*, 113(3).

LEONARD, D., AND P. COY (2012): "Alan Greenspan on His Fed Legacy and the Economy," Businessweek, August 9th.

LUCAS, R. E., N. L. STOKEY, AND E. PRESCOTT (1989): *Recursive methods in economic dynamics.* Harvard University Press.

MCCALL, J. J. (1970): "Economics of Information and Job Search," *Quarterly Journal of Economics*, 84(1), pp. 113–126.

MILGROM, P., AND I. SEGAL (2002): "Envelope Theorems for Arbitrary Choice Sets," *Econometrica*, 70(2), pp. 583–601.

PRÉKOPA, A. (1973): "Logarithmic concave measures and functions," *Acta Scientiarum Mathematicarum*, 34(1), 334–343.

RINCON-ZAPATERO, J. P., AND M. S. SANTOS (2009): "Differentiability of the value function without interiority assumptions," *Journal of Economic Theory*, 144(5), 1948 – 1964.

——— (2012): "Differentiability of the value function in continuous-time economic models," *Journal of Mathematical Analysis and Applications*, 394(1), 305 – 323.

Rockafellar, R. T. (1970): *Convex analysis*, no. 28. Princeton university press.

Smith, L. (1994): "A model of exchange where beauty is in the eye of the beholder," Discussion paper, MIT mimeo.

Topkis, D. M. (1978): "Minimizing a submodular function on a lattice," *Operations research*, 26(2), 305–321.