Optimal Employment Contracts with Hidden Search

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Abstract

In this paper I show that an optimally designed employment contract will backload compensation if the worker can summon outside competitive wage pressure through non-contractable search. The analysis is set in a random search framework, where workers search on and off the job for employment opportunities similar to that of Lentz (2010) and Bagger and Lentz (2013). The worker determines the frequency by which employment opportunities arrive through a costly choice of search intensity, which is unobserved by the firm and cannot be directly contracted upon. Firms differ in the productivity by which they employ workers. Firms compete over workers in terms of utility promises. An employment contract specifies a tenure conditional wage and search intensity path as well as a response function to possible outside offers. A renegotiation proof contract has firms competing over workers in a fashion similar to that of Postel-Vinay and Robin (2002). As in Burdett and Coles (2003) and Burdett and Coles (2010), optimal tenure conditional contracts are shown to be back loaded to discourage the worker from generating outside competitive pressure. The analysis establishes existence, uniqueness and provides characterization of the core mechanism. In addition, it is shown that contract design constraints such as minimum wages can result in ex ante rent extraction by the worker, by which I mean that firms choose to give a utility level strictly above what is required by the competition between firms. I calibrate a steady state equilibrium of the model to the favored moments in Hornstein et al. (2011) and demonstrate the model’s ability to generate wage dispersion as seen in the data even in the absence of worker heterogeneity.

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1 Introduction

In a frictional labor market, a firm may be quite certain that it can dictate poor employment terms to an employee and at least in the short run still maintain the employment relationship. However, the firm would naturally be concerned that the worker will be unhappy and go to great lengths to find an outside employer that will offer better terms. Such search is costly. In addition to the significant time and psychological effort it takes for the worker to meet other firms and to market him or herself, it is likely that the effort precludes investments in the current relationship. Also, whatever good will and other intangible assets the worker holds in relation to the current firm are likely to suffer in the process.

In this paper, I show that it is optimal for the firm to give an employment contract that backloads compensation so as to discourage the worker from bringing outside competitive pressure on the match. In doing so, the firm (at least partially) preempts the worker’s costly search effort by promising her wage growth within the contract that she would otherwise have delivered herself in the shape of outside offers.

Workers choose how intensively to search for outside job opportunities where greater intensity comes at a greater cost. Search is non-contractable and so the firm must attempt to control the worker’s search intensity indirectly through the design of the tenure conditional wage profile. The analysis is set in a random search framework, where workers search on and off the job. Firms differ in their labor productivity. If given a choice between two employment contracts, the worker will choose the contract that promises the highest lifetime utility. Thus, the analysis will adopt the shorthand that whenever firms compete with each other over a worker, they do so by issuing lifetime utility promises.

An employment contract is a history conditional specification of a wage profile, incentive compatible search intensity, and a response to outside competition should it appear. If an already employed worker meets another firm, she can choose to reveal this meeting to her current employer at which point both firms immediately learn each other’s productivities. The worker’s employment contract specifies a utility offer from her current firm conditional on the type of the outside firm. The outside firm can then submit a response to the offer. The contract contains as special cases contract posting as in Burdett and Coles (2003) and offer matching as in Postel-Vinay and Robin (2002).
Contracts are required to be renegotiation proof. The analysis demonstrates that by implication the optimal contract responds to an outside offer by matching the outside firm’s willingness to pay for the worker. If the incumbent cannot match the outside firm, it offers its own willingness to pay. Thus, competition between firms is resolved as in Postel-Vinay and Robin (2002).

Therefore, if an employed worker meets an outside firm that is willing to offer more than the value of the worker’s current contract, the meeting will result in an improvement in the value of her employment, be it with the new firm or the old. If in addition the outside firm is less productive than the current firm, the worker ends up staying with the current firm. This particular kind of event represents a simple transfer of rents from the firm to the worker but with no productive efficiency gains. That is, the worker’s incentives to search include simple rent extraction from the current employment relationship. As a result, search is inefficiently high as viewed by the two contracting parties. The optimal contract backloads wages so as to reduce this inefficiency.

The paper’s focus on dynamic employment contracts is closely related to Burdett and Coles (2003), Burdett and Coles (2010), Menzio and Shi (2010), and Lamadon (2014), and to make the problem interesting I follow their assumption of risk averse workers and imperfect capital markets.

Within a job wages are monotonically increasing in tenure and upon arrival of outside offers if revealed by the worker.\(^1\) Search intensity is monotonically decreasing in tenure and outside offer arrivals, if revealed. The worker’s lifetime utility increases monotonically over an employment spell, be it within or between jobs, but as in Postel-Vinay and Robin (2002) actual wages may decrease between jobs. Also in contrast to the standard on-the-job search model, it is possible for search intensity to increase between jobs.

In contrast to Postel-Vinay and Robin (2002) but consistent with Burdett and Coles (2010), the optimal employment contract also implies a decreasing job separation hazard in tenure because the worker’s search intensity is declining with duration and limits to the jointly efficient level at the point where the worker is extracting all the surplus from the match. In the Burdett and Coles (2010) framework the declining separation hazard is a result

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\(^1\)The framework is very flexible and as shown in a previous version of this paper, it can easily be extended to allow for human capital growth which then provides a third channel for wage growth.
of the worker turning down more outside offers as he or she moves up the wage/tenure profile, which is then associated with an increase in the incidence of inefficient job separation. It is worth noting that in contrast with the Burdett and Coles (2010) framework, all separation in this paper is efficient. Instead, search inefficiencies are related to the intensity margin.

The analysis establishes existence, uniqueness, and provides characterization of the core employment contract mechanism along with a numerical simulation to illustrate the mechanism. Subsequently, the study considers design constraints with a particular focus on rent division. The Postel-Vinay and Robin (2002) framework has the feature that employers extract all rents associated with a match in excess of the total surplus of a worker’s current match. Hence, if the worker is either unemployed or extracting full surplus in a given match, the worker receives none of the additional rents associated with meeting a more productive employer. The introduction of design constraints, in particular minimum wages, result in additional rent extraction by the worker.

The contracting framework in the paper studies ex ante rent extraction due to constraints on the contract or the shape of the utility function. The presence of a minimum wage is one example which is also studied in Flinn and Mabli (2009) in a flat wage contract Cahuc et al. (2006) and Dey and Flinn (2005) setting. A utility function that has the feature that \( \lim_{w \to \infty} u'(w) = -\infty \) will also effectively impose a minimum wage \( \underline{w} \) on the problem. Finally, the arbitrary constraint of for example a flat wage profile can also by itself result in ex ante rent extraction. In all these cases, it is possible that employers will voluntarily give the worker a strictly higher lifetime utility than what is required by the worker’s outside options. Consequently, there will be strictly positive rents to unemployed search and to search where the worker is extracting full surplus from her current match.

The contracting framework is remarkably flexible and easy to solve. It can easily accommodate important extensions such as ex ante productive worker heterogeneity and stochastically evolving firm and worker productivity. Both features that are obviously important for the empirical study of wage dynamics. While not explored in this paper, it is a promising path for future research, one which this paper has charted. The obvious alternative framework where firms post contracts as in the important work by Burdett and Coles (2003) and Burdett and Coles (2010) does not provide the same analytical simplicity and quickly becomes exceedingly complicated with the introduction of worker heterogeneity.
As in the Postel-Vinay and Robin (2002) setting, the model provides rich wage dispersion and dynamics as well as attractive worker flow patterns. In section 5 I present the steady state equilibrium of the model. I calibrate it to the favored moments in Hornstein et al. (2011) and demonstrate that if asked to, the model can fully match wage dispersion in the data even in the absence of worker heterogeneity.

The paper proceeds as follows: In Section 2, I analyze the core employment contract design problem with hidden search. Here I establish existence and uniqueness of the contract and provide characterization. The main part of the analysis is set in discrete time and the continuous time formulation is obtained as the limit when period length is taken to zero. In Section 3 I consider constraints on the design program, in particular minimum wages. In Section 4 I briefly touch on socially efficient search. Section 5 presents steady state equilibrium and calibration. Section 6 concludes.

2 The Basic Mechanism

In this section I present the core wage mechanism along with characterization, existence and uniqueness proofs. For expositional simplicity, the arguments are presented in discrete time. In the following sections I use these results to expand the analysis to a steady state equilibrium and also consider worker heterogeneity, human capital growth, savings, and design constraints. This part of the analysis is done taking the continuous limit of the discrete time environment which allows for simpler expressions in the now more complicated setting.

2.1 Environment

Time is discrete. Each period has length $\Delta$. Workers and firms are infinitely lived and all discount time according to discount factor $\beta(\Delta) = e^{-r\Delta}$. All workers are identical. Firms differ in their match output levels, which are normalized to lie in the unit interval, $p \in [0, 1]$.

Workers can be either employed or unemployed. Workers consumer their income, $w\Delta$, in a given period, the utility of which is given by increasing and concave function, $u(w)\Delta$. An unemployed worker has zero income.

Matches are created through a frictional search process. Specifically, through a costly
search choice \( \lambda \) the worker sets the probability, \( s(\Delta) = 1 - e^{-\lambda\Delta} \), that she will meet a vacancy during the period in question. Regardless of employment state, the cost of the choice of \( \lambda \) is given by increasing and convex cost function \( c(\lambda) \Delta \). A vacancy is characterized by its output level which is drawn from the cumulative offer distribution, \( \Phi(p) \), which has density \( \phi(p) \). A match ends either through an exogenous destruction probability, \( d(\Delta) = 1 - e^{-\delta\Delta} \), or if the worker quits to another firm as a result of meeting a more attractive vacancy. To save on notation in the discrete formulation of the problem, I will set up the problem where the worker is directly choosing the offer arrival probability, \( s \in [0, 1] \) at per unit of time cost \( \hat{c}(s) = \Delta c\left(-\frac{\ln(1-s)}{\Delta}\right) \).

A firm’s match output level is assumed to be time independent. The same is true for the search cost function. Generally, it will be assumed that there are no aggregate shocks and that the economy is in steady state. I make the following assumptions on the model fundamentals.

**Assumption 1.** \( u(\cdot) : \mathbb{R} \to \mathbb{R} \) is strictly increasing, strictly concave and at least twice differentiable.

**Assumption 2.** The first derivatives of the utility function are bounded, such that for any \( w \in \mathbb{R}, u'(w) \in [u'_-, u'_+] \).

**Assumption 3.** The offer distribution has no mass points and everywhere positive mass, \( \forall p \in [0, 1], 0 < \phi(p) < \infty \).

**Assumption 4.** \( c(\lambda) : \mathbb{R}_+ \to \mathbb{R} \) is strictly increasing, strictly convex and at least twice differentiable. Furthermore, \( c(0) = c'(0) = 0 \).

Firms are constant returns to scale and make decisions at the match level to maximize the expected profits of the match. Each firm posts vacancies according to some process which in steady state equilibrium results in the distribution of productivities in the vacancy pool, \( \Phi(\cdot) \).\(^2\) In section 3 I consider deviations from Assumption 2. In general, Assumption 2 is a sufficient condition that ensures that the utility function does not force the wage contract design into a corner solution. In section 3, I am specifically interested in the corner solutions.

\(^2\)This is the same approach as in Postel-Vinay and Robin (2002) and many other equilibrium analyses that estimate the offer distribution which can then be mapped back into an underlying distribution of firm heterogeneity according to the particular choice of vacancy creation process.
2.2 Employment Contract Design

The contractual environment is one of limited commitment and contracts are required to be renegotiation proof. The worker can at any point costlessly quit to unemployment. Consequently, the contract must at any point promise the worker no less than the value of unemployment. Likewise, the firm can also at any point lay off the worker into unemployment at no cost should it want to do so. Thus, the firm’s participation constraint dictates that the contract can at no point dictate negative profit value to the firm.

If an already employed worker meets another firm, the worker can choose whether or not to reveal the meeting. If the worker reveals the meeting, both firms immediately learn each other’s types. An employment contract specifies a response to the worker’s announcement of an outside meeting conditional on the type of the competing firm. A commitment to not match outside offers is contained as a special case here. In this case, the response to an outside meeting is to continue the contract as if nothing has happened, and the worker can make a choice to continue in the contract or leave for the other firm.

For simplicity, the competition between the two firms is specified such that the incumbent employer makes an offer conditional on the type of the outside firm which the outside firm can then respond to. One can allow for additional rounds of responses in the spirit of an ascending auction, but the current protocol will suffice to capture the Bertrand features of the competition.

The worker’s search intensity choice is not observable by the firm and the worker cannot commit to any particular search choice. As a result, the specified search intensity in the contract must satisfy incentive compatibility.

In line with the no borrowing or saving assumption in the simple model, the setup assumes no side payments. This rules out two obvious mechanisms that could implement an efficient outcome in some of the environments in the paper: One is the case where the worker “buys” the job with an up front payment and then receives all the rents from the match going forward. The backloading result in the paper is in a sense an imperfect implementation of this solution subject to missing credit markets and risk aversion (or other reasons why an up front payment is not feasible or attractive). The other mechanism is a bonding or “non-compete” clause in the contract, which specifies a payment from the worker to the firm.
should she move to another firm. “Non-compete” clauses are illegal in many labor markets and in others there is substantial ambiguity as to the enforceability. I assume that this type of instrument is not available to the contract design.

2.2.1 The Resolution of Competition Between Firms

The contract dictates how the firm will respond to outside competition. In the subsequent analysis, it is useful to make some initial observations about how a renegotiation proof contract must respond to outside competition.

First, define by $\Pi (V, p)$ a productivity $p$ firm’s expected discounted stream of profits from a match subject to a utility promise $V$. Denote by $\bar{V} (p) = \sup \{ V \in \mathbb{R} | \Pi (V, p) \geq 0 \}$ the firm’s willingness to pay for a worker. In section 2.5 I show that the willingness to pay is monotonically increasing in $p$. Also, define the inverse of the willingness to pay function, $\bar{p} (V)$ as the firm type that has willingness to pay $V$.

Let $V^o (\bar{V}', \sigma, p)$ specify the continuation utility of a productivity $p$ firm employment contract that has history $\sigma$ in the event the worker reveals a meeting with an outside firm that has willingness to pay $\bar{V}'$. Lemma 1 states that the optimal renegotiation proof contract matches outside offers and is history independent. The worker reveals any outside meeting where the outside firm has willingness to pay greater than the value of the worker’s current contract. Any other meeting is not revealed.

**Lemma 1.** The optimal renegotiation proof contract is characterized by $V^o (\bar{V}', \sigma, p) = \max [U, \min [\bar{V}', \bar{p} (V)]]$. The worker reveals a meeting if and only if the outside firm’s willingness to pay is greater than the continuation value of the current contract.

**Proof.** Denote by $V$ the contract’s continuation utility in the absence of an outside meeting. The employer has willingness to pay $\bar{V}$. By limited commitment it must be that $\bar{V} \geq V$. Denote by $V^o (\bar{V}')$ the contract’s utility promise in case the worker announces an outside meeting with a firm that has willingness to pay $\bar{V}'$. As mentioned above, committing to not match outside offers (which is a key defining feature of the contract posting environment) would be characterized by, $V^o (\bar{V}') = V$ for any $\bar{V}'$.

Suppose the worker meets an outside firm with willingness to pay $\bar{V}' \in [V, \bar{V}]$, that is, the outside firm can improve upon the worker’s current contract value, but cannot beat the
incumbent firm’s willingness to pay. Suppose the worker reveals the meeting. If the contract specifies \( V^o(\bar{V}') < \bar{V}' \), the outside firm would match the \( V^o(\bar{V}') \) offer and the worker would move to the new firm with this promise. This is not renegotiation proof. Upon the revelation of the outside meeting, the incumbent firm and worker would agree to renegotiate the existing contract so that \( V^o(\bar{V}') \in [\bar{V}', V] \), which will make both worker and incumbent firm weakly better off. It is this renegotiation subgame that also makes it an optimal subgame perfect strategy for the worker to reveal the outside meeting in the first place. Thus, \( V^o(\bar{V}') < \bar{V}' \) cannot be part of an optimal renegotiation proof strategy. Suppose the contract dictates \( V^o(\bar{V}') > \bar{V}' \). This cannot be part of a profit maximizing contract design since the firm will win the services of the worker with the lower utility promise, \( V^o(\bar{V}') = \bar{V}' \) and as will be shown, \( \Pi(V, p) \) is always decreasing in the relevant utility promise range in this case. Hence, an optimal renegotiation proof contract must have \( V^o(\bar{V}') = \bar{V}' \) for \( \bar{V}' \in [V, \bar{V}] \). Thus, the outcome in this case is that the worker reveals the meeting. The worker ends up staying with the current firm with utility promise \( \bar{V}' \).

Suppose the worker meets an outside firm with willingness to pay \( \tilde{V}' > \bar{V} \). By a similar argument as above, it will be an optimal subgame perfect strategy for the worker to reveal this meeting. Suppose the worker’s current contract states \( V^o(\bar{V}') < \bar{V} \). In this case the worker moves to the outside firm with a utility promise of \( V^o(\bar{V}') \). This is not renegotiation proof. Upon revealing the offer, both worker and firm would agree to a change in the contract so that \( \tilde{V}' > V^o(\bar{V}') \geq \bar{V} \). This strictly improves the workers position and leaves the firm no worse off. Consider the case where the firm attempts to help the worker extract rents from the outside firm in excess of \( \bar{V} \), that is \( V^o(\bar{V}') > \bar{V} \). It will be optimal for the outside firm to counter the offer with a promise of \( \bar{V} + \varepsilon \) where \( \varepsilon > 0 \) is arbitrarily small. The worker realizes that should he accept the incumbent firm’s offer of \( V^o(\bar{V}') > \bar{V} \), it will be immediately renegotiated down to some utility promise no greater than \( \bar{V} \) since the firm will otherwise want to lay off the worker. Hence, the worker will accept the outside firm’s offer of \( \bar{V} + \varepsilon \). Therefore, the optimal renegotiation proof contract must be such that \( V^o(\bar{V}') = \bar{V} \) for any \( \bar{V}' > \bar{V} \). And the outside firm will match this offer. The outcome is that the worker moves to the outside firm with a utility promise of \( \bar{V} \).

Suppose the worker meets an outside firm with willingness to pay \( \tilde{V}' < V \). Suppose \( V^o(\bar{V}') > V \). In this case, the worker would want to reveal the outside meeting which would then result
in an improvement in employment terms over not revealing the meeting. If the firm would somehow want to encourage the worker to search, one could imagine optimal strategies that involve this outcome, but this is not the case in the present analysis, in fact the firm’s desires are exactly the opposite. I will in the following be adopting a simplification in the analysis formulation that does not allow me to consider this case, but it will be clear from the analysis that such a strategy will be dominated by $V \geq V^o (V') \geq V'$. Next, consider the strategy $V^o (V') \leq V'$. The case where $V^o (V') < V'$ is not renegotiation proof by argument similar to the ones above and is ruled out. This leaves $V \geq V^o (V') \geq V'$. The worker will choose to not reveal the outside meeting subject to this strategy, and so it is off the equilibrium path. Subgame perfection refines to $V^o (V') = V'$.

Hence, the outcome in this case is that the worker does not reveal the meeting and the match continues with utility promise $V$. □

To summarize, the renegotiation proof contract resolves competition in a way that exactly resembles Postel-Vinay and Robin (2002). If a worker meets an outside firm that cannot improve on his current terms, the worker simply continues in the current contract and does not reveal the meeting. If the outside firm can improve on current terms but cannot match the incumbent firm’s willingness to pay, then the meeting is revealed and the worker will continue with the current firm with a utility promise equal to the outside firm’s willingness to pay. If the outside firm has willingness to pay greater than the incumbent, the worker moves to the outside firm with a utility promise equal to the incumbent firm’s willingness to pay. The option to not match outside offers is part of the choice set for the contract design but is not renegotiation proof.

2.2.2 A Simplified Contract State Space

The history of an employment contract is in its full form the path of past wages, search intensities, and outside meetings, $\sigma (t) = \{w_\tau, s_\tau, \omega_\tau\}_{\tau=0}^t$, where $\omega_\tau = (I (\text{meeting}) , V_\tau)$ is the description of whether an outside meeting was revealed and in the case of a meeting the type of the outside firm stated in terms of willingness to pay. The employment contract

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3If the worker cannot hide outside meetings some interesting considerations arise here: The firm can discourage worker search by punishing outside meetings in the case where the outside firm’s willingness to pay $V'$ is less than $V$. As long as $V^o (V') \geq V'$ such a contract is renegotiation proof. In optimal contracts with strictly positive search, the optimal choice of $V^o (V')$ is determined in a non-trivial trade off between discouraging search and consumption smoothing. Since the arrival frequency of weaker firms is firm type dependent, the instrument lies in the domain of stronger firms more so than the weak.
is a specification of history dependent wages, search intensity, and outside offer responses \( \{w(\sigma), s(\sigma), V^o(\sigma)\} \), where the renegotiation proof outside offer response function is detailed above. By the by now standard reformulation of the problem’s state space in terms of utility promises in Spear and Srivastava (1987) and Thomas and Worral (1988), the contract will instead by described by the functions \( \{w(V, p), s(V, p), V^o(V, V', p)\} \), which is the wage, search intensity and outside offer response conditional on the current utility promise and firm type and in the case of an outside offer, also the outside firm’s willingness to pay.

In the design analysis in the following, I will implement the \( V^o(V, V', p) \) outside offer response function directly into the recursive formulation of worker and firm contract valuations. Thus, the remainder of the design analysis will be solely focused on the determination of the optimally designed functions \( \{w(V, p), s(V, p)\} \). The asset equations in the following are also consistent with an interpretation where an outside offer induces contract renegotiation based on Bertrand competition between the firms along the lines of Postel-Vinay and Robin (2002). But formally, an outside offer does not induce renegotiation of the contract. Instead, it continues at pre-specified utility promise conditional on type of the outside offer and if the worker chooses to stay with the firm.

### 2.3 Worker Lifetime Utility

For now, set \( \Delta = 1 \) and in the notation suppress the dependence of the arrival rates on the length of a period. Denote by \( V_t \) the worker’s valuation of her expected discounted stream of future utilities given a current employment contract with history \( s(t) \). In the valuation of possible outside employment opportunity offers, it useful to characterize a vacancy by its willingness to pay, \( V(p) \), rather than the productivity. Define by \( F(V) \) the distribution of willingness to pay across vacancies, \( F(V) = \Phi(\bar{p}(V)) \). Furthermore, define \( \hat{F}(V) = 1 - F(V) \). With this and suppressing explicit dependence on the contract history in the notation, the worker’s valuation of her current contract with an employer with willingness
to pay \( \bar{V}(p) \) can be written as,

\[
V_t = u(w_t) - \hat{c}(s_t) + \beta dU \\
+ \beta (1 - d) \left[ s_t \int_{V_{t+1}}^{\bar{V}(p)} V dF(V) + s_t \bar{V} \hat{F}(\bar{V}) + [s_t F(V_{t+1}) + (1 - s_t)] V_{t+1} \right] \\
= u(w_t) - \hat{c}(s_t) + \beta \left\{ dU + (1 - d) \left[ s_t \int_{V_{t+1}}^{\hat{V}} \hat{F}(V) dV + V_{t+1} \right] \right\}.
\]

(1)

In period \( t \), the worker receives utility \( u(w_t) \) from wage income and takes disutility \( \hat{c}(s_t) \) from the search effort. If the worker is laid off, she receives continuation utility \( U \), which is simply the value of being unemployed. By the assumption of steady state, the value of unemployment is time independent. If her job survives, and she does not receive an outside offer, her continuation value in the employment contract is \( V_{t+1} \). In the case where she does receive an offer, denote the outside employer’s willingness to pay by \( \bar{V}' \). In the case where \( \bar{V}' \leq V_{t+1} \) the worker chooses to not reveal the meeting. If \( V_{t+1} < \bar{V}' \leq \bar{V} \) the worker will reveal the meeting and by the arguments in section 2.2.1 she receives a new contract with her current employer with an initial lifetime utility promise of \( \bar{V}' \). Finally, if \( \bar{V}' > \bar{V} \) the worker will leave her current employer and receive an employment contract with the outside firm with an initial utility promise of her old employer’s willingness to pay, \( \bar{V} \). The second line follows by integration by parts.

The implicit assumption in the above expression is that an employer will not voluntarily offer more lifetime utility than what is dictated by the direct competition between employers. In section 3, I discuss contract design constraints that can result in a violation of this assumption, in which case equation (1) is modified to reflect that job-to-job transitions can be associated with a new employment contract that has value in excess of the willingness to pay of the old employer.

An unemployed worker receives income flow normalized at zero. Given the case where employers give no more lifetime utility than strictly required by the competition between employers, there are no rents to unemployed search. Any meeting will result in an employer giving the unemployed worker a take it or leave it offer matching the value of unemployment and nothing more.\(^4\) Consequently, the value of unemployment in this case is very simply

\(^4\)The discrete setting naturally allows modifications of this argument, say, as in Burdett and Judd (1983) where the worker meets multiple employers in a given period and they have to compete with each other
equivalent to the infinitely discounted stream of utility of unemployed income,

\[ U = \frac{u(0)}{1 - \beta}. \]  

(2)

Once we move to design constraints in section 3 that result in ex ante rent extraction, equation (2) will be appropriately modified to include the now positive option value to unemployed search.

### 2.4 Incentive Compatibility

The worker’s search intensity choice is hidden, so the contract must at any point dictate an \( s_t \) that maximizes the worker’s lifetime utility going forward,

\[ s_t \in \arg \max_{s \geq 0} \left[ u(w_t) - \hat{c}(s) + \beta \left\{ dU + (1 - d) \int_{V_{t+1}}^{\bar{V}} \hat{F}(V) dV + V_{t+1} \right\} \right]. \]

Given strict convexity of the search cost function, this is a concave optimization problem and the unique maximizer satisfies the first order condition,

\[ \hat{c}'(s_t) = \beta (1 - d) \int_{V_{t+1}}^{\bar{V}} \hat{F}(V) dV. \]  

(3)

It is straightforward to show that \( s_t \) is strictly declining in \( V_{t+1} \) as long as \( f(V) = \frac{\partial F(V)}{\partial V} > 0 \) for all \( V \in [U, \bar{V}(1)] \).

### 2.5 Optimal Contract Design

For a given utility promise, the firm will design an employment contract that maximizes the future discounted profit stream of the match subject to the design constraints and the lifetime utility promise. Following Spear and Srivastava (1987); Thomas and Worral (1988); Atkeson and Lucas (1992); Menzio and Shi (2010) the firm’s employment contract design problem can be represented by a recursive formulation of the firm’s valuation of a match as a function of the lifetime utility promise to the worker at that point.

Bertrand style over the worker. It is a rather elegant way of introducing ex ante rent extraction into the setup, and the probability of meeting multiple employers in a single period is handled naturally as multiple arrivals of the Poisson arrival process. However, since the paper will be emphasizing the continuous limit of the setting and this competitive mechanism does not survive in the limit, I have chosen to ignore the possibility of multiple meetings in a single period.
For a given optimal employment contract, $C$, and given the firm’s willingness to pay, $\bar{V}(p)$, the firm’s valuation of the associated future discounted profit stream can at time $t$ be written recursively as,

$$\Pi(V_t) = p - w_t +$$

$$\beta (1 - d) \left\{ s_t \int_{V_{t+1}}^{V} \Pi'(V) dF(V) + [1 - s_t \hat{F}(V_{t+1})] \Pi(V_{t+1}) \right\}$$

(4)

$$= p - w_t + \beta (1 - d) \left\{ s_t \int_{V_{t+1}}^{V} \Pi'(V) \hat{F}(V) dV + \Pi(V_{t+1}) \right\}.$$  

(5)

where steady state has already been invoked to eliminate time dependency of $\Pi(V)$. The per period profit is $p - w_t$ and if the match survives, the firm’s continuation value associated with promising the worker a lifetime utility in the next period of $V_{t+1}$ is given by $\Pi(V_{t+1})$. The firm will deliver $V_{t+1}$ to the worker in the case where the worker does not obtain an outside offer or if the worker meets an outside firm with willingness to pay less than $V_{t+1}$ in which case the worker chooses to not reveal the offer. If the worker meets an outside firm with willingness to pay, $\bar{V}' \in [V_{t+1}, \bar{V}]$, the match survives, but the employment contract will be renegotiated to deliver $\bar{V}'$ to the worker, which has value $\Pi(\bar{V}')$ to the firm. If the outside employer has willingness to pay $\bar{V}' > \bar{V}$, the match ends. The firm’s outside value of a match is zero. The second line follows from integration by parts and that by definition, $\Pi(\bar{V}) = 0$.

I impose incentive compatibility on the employment contract design problem through the first order approach and require that any choice of $s_t$ must satisfy equation (3). Given a lifetime utility promise of $V \in [U, \bar{V}]$ and suppressing the functional dependence on $\bar{V}$ to save on notation, the firm’s valuation of an optimal employment contract is given by,

$$\Pi(V) = \max_{(w,Y,s) \in \Gamma(V)} \left[ p - w + \beta (1 - d) \left\{ s \int_{Y}^{V} \Pi'(V) \hat{F}(V) dV + \Pi(Y) \right\} \right],$$  

(6)
where for any \( V \in [U, \bar{V}] \) the set of feasible choices of the triple \((w, Y, \lambda)\) is given by,

\[
\Gamma (V) = \left\{ (w, Y, s) \in \mathbb{R}^2 \times [0, 1] \mid \right. \\
u (w) - \hat{c} (s) + \beta \left\{ dU + (1 - d) \left[ s \int_{Y}^{\bar{V}} \hat{F} (V') dV' + Y \right] \right\} \geq V \tag{7}
\]
\[
\hat{c}' (s) = \beta (1 - d) \int_{Y}^{\bar{V}} \hat{F} (V') dV' \tag{8}
\]
\[
U \leq Y \leq \bar{V} \right\}, \tag{9}
\]

where the functional dependence on \( \bar{V} \) is again suppressed. Equation (7) is the promise keeping constraint stating the contract must deliver no less than the promise of \( V \). Part of establishing consistency of the setup requires showing that this constraint is always binding. Equation (8) is the incentive compatibility constraint. Equation (9) represents the worker and firm participation constraints.

To begin the characterization of the optimal employment contract, Lemma 2 establishes existence and uniqueness of a solution to the problem in (6).

**Lemma 2.** For a given continuous and differentiable willingness to pay offer distribution \( F (\cdot) \) with support \([U, \bar{V}]\), there exist for any \( \bar{V} \in [U, \bar{V}] \) a unique solution \( \Pi (V) \) to equation (6), for any \( V \in [U, \bar{V}] \). Furthermore, if \( \Pi (V) \) is concave in \( V \) then it is also differentiable.

**Proof.** All the arguments in the proof are conditional on the firm's willingness to pay \( \bar{V} \in [U, \bar{V}] \). For notational convenience, the functional dependence on \( \bar{V} \) is suppressed. Equation (6) can also be stated in the form of the functional mapping \( T (\Pi) (V) \) by,

\[
T (\Pi) (V) = \max_{(w, Y, s) \in \Gamma (V)} \left[ p - w + \beta (1 - d) \left\{ s \int_{Y}^{\bar{V}} \Pi (V') dF (V') + [1 - s \hat{F} (Y)] \Pi (Y) \right\} \right]. \tag{10}
\]

By Blackwell's sufficient conditions, this is a contraction. The profit function is bounded above by the level,

\[
\bar{\Pi} = \frac{p}{1 - \beta (1 - d)}.
\]

\( s = 0 \) maximizes joint surplus between the worker and the firm in this simple version of the problem without ex ante rent extraction. In addition, a worker hired out of unemployment
is the smallest utility promise the firm can ever face, \( V = U \), and simply just giving the worker a flat wage \( w = 0 \) until the match ends is the cost minimizing way to meet the utility promise of \( V = U \). This will give the firm an expected profit of \( \bar{\Pi} \). The constraints in \( \Gamma (V|\bar{V}) \) for any \( V \geq U \) are more restrictive and result in lower profits than \( \bar{\Pi} \). Profits are bounded below by

\[
\Pi = \frac{p - u^{-1} (\bar{V} [1 - \beta (1 - d)] - \beta dU)}{1 - \beta (1 - d)},
\]

which is the profit level for a perpetually flat wage contract with no search at the utility promise \( V = \bar{V} \). This is the most restrictive utility promise and a feasible contract choice which may or may not be dominated by the optimal design choice.

To establish monotonicity, consider two functions \( \Pi^1 (V) \geq \Pi^0 (V) \) for any \( V \in [U, \bar{V}] \). Let \((w^0, Y^0, s^0)\) be a maximand of (10) given \( \Pi^0 (\cdot) \). In that case we get,

\[
T (\Pi^0) (V) = p - w^0 + \beta (1 - d) \left\{ s \int_{Y^0}^{\bar{V}} \Pi^0 (V) dF (V) + [1 - s \hat{F} (Y^0)] \Pi^0 (Y) \right\}
\leq p - w^0 + \beta (1 - d) \left\{ s \int_{Y^0}^{\bar{V}} \Pi^1 (V) dF (V) + [1 - s \hat{F} (Y^0)] \Pi^1 (Y) \right\}
\leq \max_{(w, Y, s) \in \Gamma (V)} \left\{ p - w^+ + \beta (1 - d) \left\{ s \int_{Y}^{\bar{V}} \Pi^1 (V) dF (V) + [1 - s \hat{F} (Y)] \Pi^1 (Y) \right\} \right\}
= T (\Pi^1) (V).
\]

The first inequality comes from the assumption \( \Pi^1 (V) \geq \Pi^0 (V) \). The second inequality follows from optimality and the fact that \((w^0, Y^0, s^0)\) must also be a part of the set of feasible choices given \( \Pi^1 \). Hence, monotonicity is satisfied. Turning to discounting, consider
an upward shift of a given function by \( a > 0 \),

\[
T(\Pi + a)(V) = \frac{\max_{(w, Y, s) \in \Gamma(V)} \left[ p - w + \beta (1 - d) \left\{ s \int_Y^V \Pi(V) dF(V) + \left[ 1 - s \hat{F}(Y) \right] \Pi(Y) + a \left[ 1 - s \hat{F}(V) \right] \right\} \right]}{\max_{(w, Y, s) \in \Gamma(V)} \left[ p - w + \beta (1 - d) \left\{ s \int_Y^V \Pi(V) dF(V) + \left[ 1 - s \hat{F}(Y) \right] \Pi(Y) \right\} + \beta (1 - d) a \right]}
\]

\[
< T(\Pi)(V) + \beta(1 - d) a
\]

for some \( \beta' \in (\beta(1 - d), 1) \). Since \( \beta(1 - d) < 1 \), such a \( \beta' \) exists and hence, discounting is satisfied. Thus, Blackwell’s sufficient conditions for a contraction are satisfied. Therefore by the contraction mapping theorem, there exists a unique fixed point to the mapping in equation (10), which is then the unique solution to the problem in equation (6).

Given concavity of the solution to (6) it is also differentiable. The proof is a straightforward application of Benveniste and Scheinkman reproduced as Theorem 4.10 in Stokey and Lucas (1989). Take the solution to (6) at some point \( V_0 \in [U, \bar{V}], \Pi(V_0) \). Let \((w_0, s_0, Y_0)\) be the maximand of the problem and let \( \tilde{V}_0 = w_0 - \hat{c}(s_0) + \beta \left\{ dU + (1 - d) \left[ s_0 \int_{Y_0}^{\bar{V}} \hat{F}(V') dV' + Y_0 \right] \right\} \). Define \( D \) as a neighborhood around \( V_0 \). Then, define the function for \( V \in D \),

\[
\tilde{\Pi}(V) = p - \tilde{w}(V) + \beta (1 - d) \left\{ s_0 \int_{Y_0}^{\bar{V}} \Pi(V) dF(V) + \left[ 1 - s_0 \hat{F}(Y_0) \right] \Pi(Y_0) \right\},
\]

where

\[
\tilde{w}(V) = u^{-1} \left( \tilde{V}_0 + (V - V_0) + \hat{c}(s_0) - \beta \left\{ dU + (1 - d) \left[ s_0 \int_{Y_0}^{\bar{V}} \hat{F}(V') dV' + Y_0 \right] \right\} \right).
\]

Sufficient conditions for existence of \( \tilde{w}(V) \) can be something like \( \lim_{w \to -\infty} u(w) = -\infty \) and \( \lim_{w \to \infty} u(w) = \infty \), but is much stronger than what is needed anywhere else in the proofs. By construction, the wage \( \tilde{w}(V) \) in combination with search intensity \( s_0 \) and continuation promise \( Y_0 \) will give the worker exactly current lifetime utility promise \( V \). Given concavity
of $u(\cdot)$ this implies that $\tilde{\Pi}(V)$ is a concave function in $V$ and by Assumption 1, it is differentiable in $V$. Furthermore, since $(\tilde{w}(V), Y_0, s_0) \in \Gamma(V)$, by optimality $\tilde{\Pi}(V) \leq \Pi(V)$ and $\tilde{\Pi}(V_0) = \Pi(V_0)$. Hence, by Benveniste and Scheinkman it follows that $\Pi(V_0)$ is differentiable at $V_0$. □

Lemma 3 provides the key first conditions that the optimal contract must satisfy.

**Lemma 3.** Assume the profit function $\Pi(V)$ is differentiable. Under assumptions 1-4, for a given willingness to pay $\bar{V} \in [U, \bar{V}(1)]$ such that $\Pi(\bar{V}) = 0$, an optimal contract given by wages $w(V)$, search intensity $s(V)$, and next period’s lifetime utility promise $Y(V) \in [U, \bar{V}]$ must for any $V \in [U, \bar{V}]$ satisfy the conditions,

$$\nu(V) = \frac{1}{w'(w(V))} \quad (11)$$

$$s(Y(V)) = s' \left( \beta (1 - d) \int_V^{\bar{V}} \hat{F}(V') \, dV' \right) \quad (12)$$

$$\mu(Y(V)) = -\beta (1 - d) \int_Y^{\bar{V}} \Pi'(V') \hat{F}(V') \, dV' \quad (13)$$

$$\nu(V) + \Pi'(Y(V)) = -\Psi(Y(V)) \quad (14)$$

$$\Psi(Y(V)) = \frac{\mu(Y(V)) \hat{F}(Y(V))}{1 - s(Y(V)) \hat{Y}(Y(V))} \quad (15)$$

where $\nu(V)$ and $\mu(V)$ are the Lagrange multipliers on the (7) and (8) constraints, respectively. By Assumption 2, the promise keeping constraint is always binding, that is, $\nu(V) > 0$ for all $V \in [U, \bar{V}]$. In addition, the incentive compatibility constraint is binding for any $Y(V) < \bar{V}$ but is not binding for $Y(V) = \bar{V}$, that is $\mu(Y(V)) \geq 0$ with strict inequality for $Y(V) < \bar{V}$ and equality for $Y(V) = \bar{V}$.

**Proof.** Equations (11)-(14) are the first order conditions of the Lagrangian associated with the problem in (6) with respect to $w$, $s$, and $Y$, respectively. By assumption 2, the promise keeping constraint is always strictly binding, $\nu(V) > 0$, since the first derivatives of the utility function are bounded. The incentive compatibility constraint (12) implies that $s(Y(V)) > 0$ for all $Y(V) \in [U, \bar{V})$ and by equation (13) and Assumption 4 it follows that $\mu(Y(V)) > 0$ for all $Y(V) \in [U, \bar{V})$. For $Y(V) = \bar{V}$, and since by construction $\Pi(\bar{V}) = 0$ it follows that $s$ does not enter into the profit function. At zero profits, the firm is indifferent whether the
match ends soon or later due to worker search. Hence, the firm will want to use the worker’s search intensity only to create slack in the promise keeping constraint, which is perfectly aligned with the worker’s incentive compatibility constraint of maximizing lifetime utility. Hence, the incentive compatibility constraint (8) is not a binding constraint in this case, \( \mu (V) = 0 \).

Proposition 1 establishes that \( \Pi (V) \) must be strictly decreasing and strictly concave in \( V \) over the support, \( V \in [U, \bar{V}] \). In order to establish concavity, the proof makes the sufficient condition that the second derivative of the cost function be decreasing in the search intensity choice, as summarized in assumption 5:

**Assumption 5.** For any \( s_1 > s_0 \geq 0 \), the search cost function is such that \( \hat{c}'' (s_1) \leq \hat{c}'' (s_0) \).

The assumption is sufficient (but not necessary) to ensure that \( \Psi (\cdot) \) is decreasing which is by itself stronger than necessary to establish concavity. Assumption 5 is somewhat heavy handed, but it has the advantage of being simple and it provides a sufficient condition for the nice characteristic that \( \mu (V) \) will be everywhere decreasing in \( V \), that is, the incentive compatibility constraint will be less binding the higher the utility promise.\(^5\) With concavity established, Proposition 1 provides the key result that there exists a unique optimal employment contract, that it backloads wages and that lifetime utility is increasing in tenure as it goes toward the firm’s willingness to pay.

**Proposition 1.** Given assumptions 1-5, there exists for any productivity \( p \in [0, 1] \) firm a unique profit function \( \Pi (V|\bar{V} (p)) \) as defined in equation (6) where \( \bar{V} (p) \) is the firm’s willingness to pay. The profit function is differentiable, strictly decreasing and strictly concave over the support \( V \in [U, \bar{V} (p)] \), with \( \Pi (\bar{V} (p) |\bar{V} (p)) = 0 \). The willingness to pay is given by, \( \bar{V} (p) = \left[ u (p) + \beta dU \right] / \left[ 1 - \beta (1 - d) \right] \). Wages are backloaded, \( w (V) \leq w (Y (V)) \), with strict inequality for \( V < \bar{V} (p) \).

**Proof.** The proof uses the corollary to the contraction mapping theorem that if a contraction mapping \( T : X \rightarrow X \) maps functions in the subset \( S \subseteq X \) into a subset \( S' \subset S \), then the

\(^5\)A third order derivative condition also figures prominently in Rogerson (1985) and also in relation to establishing concavity/convexity. However, the contexts and mechanisms are quite different and seem to shed little light on each other. In addition, simulations suggest that Assumption 5 is far from necessary in establishing concavity. Examples are available upon request.
unique fixed point $v = Tv$ must belong to $S'$, that is $v \in S'$. In this case $S$ will be the set of weakly decreasing and weakly concave functions and $S'$ will the be set of strictly decreasing and strictly concave functions and $X$ is the set of continuous and bounded functions.

For a given willingness to pay $\bar{V}$, the profit function $\Pi (V)$ is the unique fixed point to the contraction mapping in equation (10). Take some decreasing and concave function $\Pi (V)$ over the support $V \in [U, \bar{V}]$. Furthermore, assume $\Pi (V)$ is differentiable. In this case, the solution to the maximization problem on the right hand side of equation (10) is characterized by equations (11)-(15) in Lemma 3. It is straightforward to show that the second order sufficient conditions for a concave problem are also satisfied in this case, so the first order equations define a unique maximizer. In addition, by the envelope theorem, it immediately follows that,

$$\frac{\partial T (\Pi) (V)}{\partial V} = -\nu (V),$$

where $-\nu (V)$ is the Lagrange multiplier on the promise keeping constraint as defined in Lemma 3. Thus, it follows that,

$$\frac{\partial T (\Pi) (V)}{\partial V} = \frac{-1}{u' (w (V))} < 0,$$

where the strict inequality follows from Assumption 2. Hence, $T (\Pi) (V)$ is strictly decreasing in $V$. In addition it must be that $T (\Pi) (V)$ is strictly concave, that is $\partial T (\Pi) (V) / \partial V$ is strictly decreasing in $V$. Assume to the contrary that there exist some $U \leq V_0 < V_1 \leq \bar{V}$ such that $\partial T (\Pi) (V_0) / \partial V_0 \leq \partial T (\Pi) (V_1) / \partial V_1$. By equation (16) this implies $w (V_0) \geq w (V_1)$. Since the promise keeping constraint is always binding the continuation utility promises must satisfy,

$$V = u (w (V)) + S (Y (V)) + \beta (1 - d) Y (V) + \beta d U,$$

where $S (V) = \max_s [\beta (1 - d) \int_V^{\bar{V}} \hat{F} (V') dV' - \hat{c} (s)]$ is strictly decreasing in $V$ by Assumptions 3 and 4. By the promise keeping constraints it follows that it must be that $Y (V_0) < Y (V_1)$. By equations (14) and (16) the continuation utility promise must satisfy,

$$\frac{\partial T (\Pi) (V)}{\partial V} - \Pi' (Y (V)) = \Psi (Y (V)).$$
From this one can make the following series of arguments,

\[
\frac{\partial T(\Pi)(V_0)}{\partial V_0} = \Psi(Y(V_0)) + \Pi'(Y(V_0)) \\
\geq \Psi(Y(V_0)) + \Pi'(Y(V_1)) \\
> \Psi(Y(V_1)) + \Pi'(Y(V_1)) \\
= \frac{\partial T(\Pi)(V_1)}{\partial V_1},
\]

where the first inequality follows from concavity of \(\Pi(V)\). The second inequality comes from Assumptions 3 and 5. Assumption 3 ensures that there is positive mass over all of the offer distribution support and so as the continuation utility promise strictly increases, the probability that the worker receives a better offer strictly declines. Consequently, the worker will search strictly less. Assumption 5 is sufficient to guarantee that a strict increase in the continuation utility promise is associated with a strictly less binding incentive compatibility constraint in the sense that \(\mu(Y(V))\) is strictly declining in \(Y(V)\). In total this implies that \(\Psi(Y(V))\) is strictly declining in \(Y(V)\). Hence, one obtains that \(\partial T(\Pi)(V_0)/\partial V_0 > \partial T(\Pi)(V_1)/\partial V_1\) which is a contradiction. Hence, it must be \(T(\Pi)(V)\) is strictly concave over the support \(V \in [U, \bar{V}]\).

By the Benveniste and Scheinkman argument in the proof of Lemma 2, as long as \(T(\Pi)(V)\) is concave it is also differentiable. By the corollary to the contraction mapping theorem, since \(T(\Pi)\) maps decreasing and concave functions into strictly decreasing and strictly concave functions, the fixed point must be strictly decreasing and strictly concave because the set of weakly increasing and weakly concave functions is a closed subset of the set of bounded and continuous functions. By the fact that concavity is mapped into concavity always, it follows that differentiability is also maintained in the iteration of the mapping as it converges to the fixed point. Therefore it must be that the profit function \(\Pi(V)\) defined in (6) must be strictly decreasing, strictly concave, and differentiable over the support \(V \in [U, \bar{V}]\).

By construction \(\Pi(\bar{V}) = 0\). Furthermore, by equation (14) the continuation utility promise at the willingness to pay satisfies that \(\Pi'(\bar{V}) = \Pi'(Y(\bar{V})) \Rightarrow Y(\bar{V}) = \bar{V}\). By the
incentive compatibility constraint, $\lambda(\bar{V}) = 0$, which gives,

$$0 = \Pi(\bar{V}) = p - w(\bar{V}) + \Pi(\bar{V})$$

$$\downarrow$$

$$w(\bar{V}) = p.$$ 

Hence, from the always binding promise keeping constraint, it follows that,

$$\bar{V}(p) = u(p) + \beta(1 - d)\bar{V} + \beta dU$$

$$\uparrow$$

$$\bar{V}(p) = \frac{u(p) + \beta dU}{1 - \beta(1 - d)}.$$ 

Therefore, the willingness to pay is strictly increasing in $p$. Hence, define $\hat{V} = \bar{V}(1)$, and in combination with Assumption 3 the conditions in Lemma 2 for existence and uniqueness are satisfied.

For the fixed point $\Pi(V) = T(\Pi)(V)$ it must be that,

$$\Pi'(V) - \Pi'(Y(V)) = \Psi(Y(V)) \geq 0,$$

which by $\Pi$ strictly decreasing and strictly concave implies that $\bar{V} \geq Y(V) \geq V$ for all $V \in [U, \bar{V}]$ with strict inequalities for $V < \bar{V}$ and equality for $V = \bar{V}$. Furthermore, again by concavity, wages are strictly increasing in the lifetime utility promise, which delivers that $w(V) < w(Y(V))$ for any $V < \bar{V}$ and $w(\bar{V}) = w(Y(\bar{V})) = p$. That is, wages are strictly increasing for utility promises below the willingness to pay. The optimal contract promises increasing lifetime utility to the point of delivering exactly the firm’s willingness to pay after which it is flat. The contract asymptotes to the willingness to pay in that $\bar{V} > Y(V) > V$ for any $V < \bar{V}$ by the strict concavity of the profit function and by the fact that $\Psi(\bar{V}) = 0$.  

By Proposition 1 it follows that $w(V)$ is increasing in the utility promise. For lifetime utility promises strictly less than the firm’s willingness to pay At the willingness to pay the worker is extracting full surplus which is delivered in a flat wage profile, $w(\bar{V}(p)) = p$ with $Y(\bar{V}(p)) = \bar{V}(p)$ and $s(\bar{V}(p)) = 0$. 

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2.6 Savings

The framework assumes no savings and borrowing, which simplifies the analysis considerably. Since an obvious solution to the moral hazard problem in the paper is for the worker to buy the match up front (which is ruled by the lack of credit markets), it is not surprising that the firm would adopt a more aggressive backloading strategy if the worker can smooth consumption by use of already accumulated savings. In the extreme, if the worker has enough savings, the optimal backloading strategy would resemble an up front payment for the job and then quickly settle into a flat wage scheme equalling the output of the match. This would resolve the inefficiency that results from the worker searching to extract rents from the current relationship. However, it rests on the presumption that the worker has made past consumption sacrifices to accumulate savings. If indeed the worker were to receive rents from eliminating or reducing the inefficiencies in future relationships, the worker would be motivated to make such consumption sacrifices to accumulate savings. However, in the baseline version of the model the rents go to the firm, and as such, the worker has no incentive to accumulate savings for the purpose of eliminating inefficiencies due to hidden search.

In section 3, the analysis will consider wage mechanism design constraints such as minimum wages that will result in ex ante rent extraction by the worker. A bargaining setup as in Calhuc et al. (2006) also has the feature that the worker can extract rents from a future match over and above the total match surplus in the current match. In those setups, the worker will receive some fraction of the increased rents due to accumulated savings. However, only in the case where the worker receives all the rents from a future match will savings not be under-accumulated.

2.7 The Continuous Time Limit

The continuous time limit of the model and its solution are obtained by taking the limit \( \Delta \to 0 \) for the results above. In this case, the design problem can be written as,

\[
[r + \delta] \Pi (V) = \max_{(w, V, \lambda) \in \Gamma(V)} \left[ p - w + \lambda \int_{V}^{V'} \Pi' (V') F (V') dV' + \Pi' (V) \dot{V} \right],
\]
where the set of feasible choices $\Gamma (V)$ is given by,

$$\Gamma (V) = \left\{ \left( w, \dot{V}, \lambda \right) \in \mathbb{R}^3 \mid u (w) - c (\lambda) + \delta U + \lambda \int_{V}^{\hat{V}(p)} \hat{F} (V') dV' + \dot{V} \geq (r + \delta) V \right\} \quad (18)$$

$$c' (\lambda) = \int_{V}^{\hat{V}(p)} \hat{F} (V') dV' \quad (19)$$

Lemma 4 provides the continuous time expressions that the solution must satisfy. The lemma is left without proof since it is a simple restatement of results in Lemma 3 and Proposition 1 taking $\Delta \to 0$. In the statement of the results, the lemma makes the assumption that the second derivative of the profit function exists. If not, the law of motion $\dot{V} (V)$ defined in equation (22) is stated in terms of $\dot{\Pi}' (V)$.

**Lemma 4.** Let the unique solution to the design problem in equation (6) in the continuous time limit, $\Delta \to 0$, be represented by the wage and search intensity functions, $w (V)$ and $\lambda (V)$ respectively, as well as a law of motion of the utility promise, $\dot{V} (V) = \lim_{\Delta \to 0} [Y (V) - V] / \Delta$. The contract is for a given utility promise $V \in [U, \hat{V} (p)]$ characterized by,

$$\Pi' (V) = \frac{-1}{u' (w (V))} \quad (20)$$

$$c' (\lambda (V)) = \int_{V}^{\hat{V}} \hat{F} (V') dV' \quad (21)$$

$$\dot{V} (V) = \frac{\hat{F} (V) \int_{V}^{\hat{V}} \Pi' (V') \hat{F} (V') dV'}{c'' (\lambda (V)) \Pi'' (V)} \quad (22)$$

$$(r + \delta) V = u (w (V)) - c (\lambda (V)) + \delta U + \lambda \int_{V}^{\hat{V}} \hat{F} (V') dV' + \dot{V} (V) \quad (23)$$

$$(r + \delta) \Pi (V) = p - w (V) + \lambda \int_{V}^{\hat{V}} \Pi' (V) \hat{F} (V') dV + \Pi' (V) \dot{V} (V) \quad (24)$$

$$U = \frac{u (0)}{r} \quad (25)$$

$$(r + \delta) \tilde{V} (p) = u (p) + \delta U, \quad (26)$$

where the willingness to pay distribution is defined by $F (V) = \hat{F} (V^{-1} (V))$.

**2.8 A simple numerical example**

To illustrate the details of the optimal contract the following provides a numerical solution of the model contracts. The details of the numerical solution of the model are discussed in
the Appendix. Make the following specifications.

$$u(w) = \frac{1 - \exp(-\alpha w)}{\alpha}$$ \hspace{1cm} (27)

$$c(\lambda) = \frac{(c_0 \lambda)^{1+c_1}}{1+c_1}. \hspace{1cm} (28)$$

Assume $\Phi(\cdot)$ is a beta distribution with parameters $(\beta_0, \beta_1)$.

Table 1 provides the parameterization of the model. In section 5.1 I calibrate the model to fit key stylized labor market outcomes. The current parameterization is made primarily for the purpose of illustrating key mechanisms in the model. The choice of $\alpha$ implies that at full rent extraction with the median firm in the offer distribution, the coefficient of relative risk aversion is approximately 3. The unemployed income flow is normalized at zero. It is therefore not surprising that negative wages can arise in the setting. In section 3.1, I solve the model subject to a constraint of non-negative wages.

<table>
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<th>$\alpha$</th>
<th>$c_0$</th>
<th>$c_1$</th>
<th>$\beta_0$</th>
<th>$\beta_1$</th>
<th>$r$</th>
<th>$\delta$</th>
</tr>
</thead>
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<td>0.25</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0.05</td>
<td>0.1</td>
</tr>
</tbody>
</table>

In Figure 1, I show the parameterized offer distribution and the firm type conditional willingness to pay. For reference, the worker’s valuation of unemployment in this parameterization is $U = u(0)/r = 0$. To illustrate the workings of the basic contract, in Figure 2, I show the contract for the median firm in the offer distribution as a function of the utility.
Note: The median firm has productivity $p = 0.392$.

promise state. A worker that is hired directly out of unemployment by the firm will start the contract at utility promise $V = 0$. As can be seen by the upper right hand panel, the utility promise will increase quickly in the duration of the employment spell. The initial wage rate is equivalent to the worker paying roughly 10 times the annual revenue rate to the firm, which on the face of it seems like a rather extreme contract. However, at the same time, the change in the utility promise is roughly $\dot{V}(0, p) \approx e^{25}$, so the worker will not be subject to the low wage rate for long. The profit value function is shown in the upper left panel. they are monotonically decreasing in the utility promise level reaching zero at the willingness to pay of $\bar{V}(p) = 0.79$. Wages are monotonically increasing in the utility promise and the search intensity is monotonically decreasing. If the worker is offered full surplus extraction with the firm, the search intensity is $\lambda(\bar{V}(p), p) = 0$ and the wage is $w(\bar{V}(p), p) = p$.

Given the behavior of the contract for low utility promises, it is more useful to view the contract as a function of duration, which is shown in Figure 3, which shows the evolution of
an employment relationship with a worker who is hired directly out of unemployment. The graphs show the evolution of the contract after one month of employment and forward. The black solid lines show the evolution of the employment relationship in the hypothetical case where the worker never receives an offer from a firm with a willingness to pay in excess of the worker’s current utility promise. The lighter green lines show the average realization of the employment relationship taking into account the firm’s response to arrivals of outside offers, conditional on those outside offers being such that the worker stays with the firm. Specifically, the black lines are obtained by solving for,

\[ V(t, V(0)|p) = V(0) + \int_0^\tau \dot{V}(V(\tau|p), p) \, d\tau, \]

where the notation has suppressed the dependence on the firm type. The average change in the utility promise including the possible arrival of outside offers that trigger renegotiation but not a quit to other firms is given by,

\[ \dot{V}_{\text{tot}}(V, p) = \dot{V}(V, p) + \lambda(V, p) \int_V^{V'} [V' - V] \, dF(V'). \]

Hence, solving for the evolution of the employment relationship subject to survival of the relationship and possible response to outside offers is done by solving,

\[ V_{\text{tot}}(t, V(0)|p) = V(0) + \int_0^\tau \dot{V}_{\text{tot}}(V_{\text{tot}}(\tau|p), p) \, d\tau. \]

Hence, the black lines in Figure 3 are based on solving for \( V(t|p) \) and the green lines are based on the solution for \( V_{\text{tot}}(t|p) \).

The evolution of the utility promise is shown in the upper left panel. The utility promise quickly increases from its start at \( V(0) = U = 0 \) to roughly \( V(1/12|p) \approx 0.71 \) within the first month (not shown). All the interesting variation in the contract is from this utility promise level and up to the willingness to pay of \( \bar{V}(p) = 0.797 \). As can be seen, in expectation outside offers will make the utility grow faster in the contract than what is directly promised in the contract specified growth rate.

The search intensity is shown in the upper right panel which is decreasing in the duration of the spell in a reflection of the increasing utility promise in duration. The decreasing search intensity manifests itself in a decreasing monthly separation hazard, shown in the lower left panel and the associated survivor function is given in the lower right panel. For the given
Figure 3: Unemployed worker’s contract with median firm type

Note: The contract is shown from the end of the first month and forward. The first observation for the monthly earnings are the accumulated earnings during the first month. The solid black lines show the evolution of the contract based on the utility promise evolution in equation (29) which assumes no outside offers that may trigger renegotiation. The green lines are based on the utility promise evolving according to equation (30), which is the case where outside offers may trigger renegotiation of the contract.

Parameterization, roughly 6.5% of workers hired out of unemployment with the firm will have left the firm within the first month, the vast majority to go to other firms. As the employment spell progresses, the separation hazard declines and will asymptote to the level
implied by the exogenous layoff rate. Of course, it is a solid empirical fact that separations rates are decreasing in match duration. Furthermore, Bagger and Lentz (2013) document that this relationship is stronger for lower rank firms than higher rank firms. In fact, in the model at hand, the separation rate will be duration independent for the highest ranked firm. In general, the relationship between duration dependence strength and firm rank need not be monotonic in the model, but properly parameterized it delivers a negative relationship as seen in the data.

The middle left panel shows the wage rate and the right middle panel shows monthly earnings. It is seen that even though the wage rate is very low early on, the growth rate in wages is so fast that the accumulated earnings in the first month are only moderately negative. Monthly earnings are increasing in duration reflecting the backloading of the contract.

Within an employment relationship, earnings grow for two reasons: The contract has committed to an increasing wage profile, and outside offers may cause wage to jump up. In the given parameterization, the arrival of outside offers and the associated response by firms has a fairly limited impact. One can calculate the value of the expected discounted earnings within the job conditional on its survival absent wage growth due to outside offers,

\[ E(0) = \int_0^\infty e^{-rt} w(V(t, 0), p) dt, \]

and the equivalent for the case where outside offers may arrive to increase wages,

\[ E_{\text{tot}}(0) = \int_0^\infty e^{-rt} w(V_{\text{tot}}(t, 0), p) dt. \]

In this particular case, one finds that \[ \frac{E_{\text{tot}}(0) - E(0)}{E_{\text{tot}}(0)} = 0.044, \] that is, discounted lifetime income would be 4.4% lower in the absence of arrivals of outside offers to force the firm to respond to keep the worker.

In Figure 4 I show earnings, match survival and probability of contract renegotiation by firm type for employment relationships with workers that are hired directly out of unemployment. Disregarding the behavior of the least productive firms, it is seen that earnings both the first month and the first year are increasing in firm type (upper left and right panels). This is certainly an attractive feature, but it represents a significant departure from the standard Postel-Vinay and Robin (2002) setup where the lowest observed wage belongs
to the unemployed worker who has just been hired by the most productive firm. Any firm that hires an unemployed worker only has to match the value of unemployment, and since the more productive firms have greater future wage growth potential, they can offer a lower initial wage. The same mechanism exists in this model. The lowest offered wage rate in
the economy is indeed given by the most productive firm to the worker just hired out of unemployment. But the combination of a concave utility function and search intensity implies that wages grow so much faster in the more productive firms that even at the high measurement frequency of a month, more productive firms give greater initial earnings than less productive firms. The concave utility function modifies how low the wage rate goes in the more productive firms, and the variable search intensity means that the more productive firms have increased incentive to grow wages fast.

In principle, if the arrival rate of offers is sufficiently fast in Postel-Vinay and Robin (2002), a similar picture would result in that setting. However, the exogenously given offer arrival rate in their framework is strongly tied down by the observed job-to-job transition rate in the data. The offer arrival rate consistent with data will in the Postel-Vinay and Robin (2002) typically not be big enough to allow for a positive relationship between the initial earnings of a recently unemployed worker and firm type such as in Figure 4. In the endogenous search intensity setting, wage growth is driven in part by the worker’s threat to search if the firm does not grow wages fast enough. That means that high wage growth is not necessarily tied to a high offer arrival rate. Rather, it is tied to the threat of a high offer arrival rate. Thus, the model can produce reasonable looking first year match survival rates as in the lower left panel, while still maintaining a sufficiently strong threat of outside competition in the more productive firms so as to generate high initial monthly wages. The bottom panel in Figure 4 further illustrates this point showing the monthly wage profile of a worker hired directly from unemployment into either the 20th, 50th, or 80th percentile firm in the offer distribution. At any point, the more productive firm pays more.

The non-monotonicities in Figure 4 are tied to a stark feature of the search choices in the basic model where the profit function is strictly decreasing in the utility promise everywhere: Workers that are currently extracting all surplus from their current relationships have no incentive to search. In particular, that means that an unemployed worker searches no more than any employed worker. In fact in the continuous firm type model since it is a zero mass event for an employed worker to extract exactly all the surplus from a relationship, it is the unemployed worker who searches the least of all workers in the economy. While it is not unreasonable to introduce a level of “free” search in the model so as to allow workers who are not searching to still have a positive offer arrival rate, most such implementations would
still have the feature that the unemployed workers have lower offer arrival rates than any employed worker.

If on the other hand, a worker can extract rents from an outside firm in excess of the value of her current match, workers that are currently receiving all the rents from their current matches have incentive to search. Cahuc et al. (2006) allow for ex ante rent extraction through a wage bargaining mechanism. Lentz (2010) and Bagger and Lentz (2013) use this feature in an endogenous search intensity setting. Conceptually, one may well implement the same bargaining mechanism in this model. It comes with a technical complication since the bargaining must now respect the curvature of the firm’s profit function, which is endogenous to the model solution. Thus, the approach comes with an added fixed point search, but it is likely a manageable one. As it turns out, natural constraints on the wage profile design can by themselves result in ex ante rent extraction. The next section places particular emphasis on the lower wage bound constraint.

The middle right panel shows the frequency of outside offer matching within the first year of a worker hired directly out of unemployment. It is worth mentioning that in a substantial fraction of firms, outside offer matching is a rare event. Within the first year, the median firm matches outside offers only for about 15% of its workers recently hired out of unemployment. Outside offer matching is less frequent in subsequent years and for workers hired from other firms. Thus, if surveyed about wage determination mechanisms, human resource managers in many low rank firms could well respond that they do not match outside offers, not as a statement of commitment to such a practice, but as a statement of fact.

3 Design Constraints

It is noteworthy that in the core problem, the worker’s lack of ability to commit ex ante to a search intensity level does not by itself result in any rent extraction by the worker. Any rents that flow to the worker are purely a result of the offer matching process. For example, an unemployed worker has in the core problem no gains from search as a future employer will offer a contract that exactly matches the value of unemployment, and no more. The same is true for any worker who is currently extracting full rents from a given employment relationship. This changes with the imposition of constraints on the choice of wage path in
the contract. In the following, I consider two constraints: (1) A minimum wage, that is the wage must at any point in time $t$ be greater than some level, $w_t \geq w$, and (2) a constraint on the change in wages, specifically impose that the wage profile must be flat, $\dot{V}_t = 0$.

The design constraints that are considered in this section have the implication that the promise keeping constraint is no longer necessarily binding for all utility promises, that is, it is possible that $\nu(V) = 0$ for some $V \in [U, \bar{V}]$. The recursive formulation of the contracting problem of course relies on the utility promise constraint to be binding. Hence, care must be taken as the analysis considers modifications to the set of feasible contracts. As it turns out, the optimal contract will have the feature that the utility promise constraint is binding for all periods in the contract following the initial period. It is at the very beginning of the contract that the firm may choose to voluntarily give the worker more utility than what was required to win the worker’s services. Thus, the worker’s valuation of search is modified since the worker may receive more lifetime utility from an employer than what is dictated by the Bertrand competition between firms. For this purpose, define the object $\underline{V}(p)$ as the least lifetime utility an employer will want to give a worker irrespective of the worker’s outside option. Since the worker always has unemployment as the outside option, if that option is binding, set $\underline{V}(p)$ equal to the value of unemployment, $\underline{V}(p) = \max(\underline{V}(p), U)$.

For expositional purposes, also define the minimum utility promise in terms of a firm’s willingness to pay rather than its productivity, $\bar{V}(\bar{p}) = \bar{V}^{-1}(p)$.

With this object, the worker’s expected lifetime utility in equation (1) is modified as follows,

$$V_t = u(w_t) - \check{c}(s_t) + \beta dU + \beta (1 - d) \left[ s_t \int_{V_{t+1}}^{\bar{V}} VdF(V) + s_t \int_{\bar{V}}^{V(1)} \max \left[ \bar{V}, \bar{V}(V) \right] dF(V) + \left[ s_t F(V_{t+1}) + (1 - s_t) \right] V_{t+1} \right],$$

which can be rewritten as,

$$V_t = u(w_t) - \check{c}(s_t) + \beta dU + \beta (1 - d) \left[ s_t \int_{V_{t+1}}^{\bar{V}} \bar{V}(V) dV + \int_{\bar{V}}^{V(1)} \max \left[ 0, \bar{V}(V) - \bar{V} \right] dF(V) \right] + V_{t+1}. \quad (31)$$

---

6In the previous section, the profit function is monotonically decreasing in the utility promise and so $\underline{V}(p) = -\infty$ absent the convention of just setting it equal to the worker’s participation constraint level.
Consequently, the incentive compatibility constraint becomes,

\[
\hat{c}' (s_t) = \beta (1 - d) \left[ \int_{V_{t+1}}^{\bar{V}} \tilde{F} (V) dV + \int_{\tilde{V}}^{V^{(1)}} \max \left[ 0, \tilde{V} (V) - \tilde{V} \right] dF (V) \right].
\] (32)

The modified incentive compatibility constraint highlights that the worker’s access to hidden search may lead to ex ante rent extraction given particular constraints on the wage contract. Thus, even though firms make take-it-or-leave-it offers to unemployed workers, there may still be strictly positive returns to unemployed search (and more generally, search subject to full rent extraction in current match). By the same token, the jointly efficient search within a given match may not be zero, since the worker may be able to extract rents from an outside firm in excess of the maximal rents of the current relationship. This is in contrast to the standard Postel-Vinay and Robin (2002) setup, where all rents to match creation fall to the firms.

Flinn and Mabli (2009) also argue the issue of ex ante rent extraction associated with minimum wages in a Cahuc et al. (2006) setup with assumed flat wage profiles and exogenous offer arrival rates. Common to their analysis and to the one in this paper is the firm expectation that its workers will on occasion meet outside employers and the subsequent contract renegotiation will result in either wage increases or separation. In the Cahuc et al. (2006) paper, the flat wage contract is then set so as to match whatever match surplus division is dictated by the Bertrand competition and wage bargaining at the outset of the match. In Flinn and Mabli (2009) this wage may be below the minimum wage in which case wages are set as a corner solution at the minimum wage and the worker ends up with a greater match surplus share than that dictated by the competitive pressures in combination with the standard bargaining.

This has similarities to the results in the analysis at hand, but the presence of risk aversion and endogenous search accentuates the relevance of a lower wage bound: The concave utility function provides a natural interpretation of the setup where the wage mechanism design translates to the firm choosing consumption profiles and observable savings levels for the worker and consequently the lower wage bound translates into a lower consumption bound, which can be justified in terms of subsistence level constraints. In addition, the incentive compatibility constraint interacts with the lower wage bound so as to accentuate its importance: In practical applications of the model, the worker’s implicit threat to search can
be a considerably stronger force for backloading wage profiles than a constant offer arrival rate across firms, and consequently it increases the potential importance of constraints on how low the initial wage levels can be. It is an interesting aspect of the analysis that the inability to commit to a search intensity level does not give the worker any match rents in the baseline case in the model. But this is because the firm can simply drop the current wage freely with no adverse effects on behavior. The minimum wage limits this mechanism and sets the stage for the result that the worker’s inability to commit to a particular search intensity effectively delivers rents to the worker.

3.1 Minimum wages

A minimum wage can be imposed on the problem in the form of \( w_t \geq w \) or it can be a consequence of a utility function specification that violates assumption A2. A utility function that has the characteristic that \( \lim_{w \to w^+} u'(w) = -\infty \) will have a similar impact on the optimal contracting problem as the simple minimum wage constraint as above. In order to show the more general features of a lower wage bound constraint, I will maintain assumption A2, but now over the support \( w \in [\underline{w}, \infty) \), and let minimum wage constraints be expressed in the form of \( w_t \geq w \). In most of the following notation, I suppress that value and policy function depend on \( p \). In this case, the firm’s optimal contract design problem is still given as in equation (6), but the constraint set is modified by,

\[
\Gamma (V|\tilde{V}) = \left\{ (w, Y, s) \in \mathbb{R}^2 \times [0, 1] \right\} \\
\left. u(w) - \hat{c}(s) + \beta dU + (1 - d) \beta \left\{ Y + s \left[ \int_{Y}^{\tilde{V}} \tilde{F}(V) dV + \int_{\tilde{V}}^{V^{(1)}} \max \left[ 0, \tilde{V} - V \right] dF(V) \right] \right\} \right\} \geq V 
\]

(33)

\[
\frac{\hat{c}'(s)}{\beta (1 - d)} = \int_{Y}^{\tilde{V}} \tilde{F}(V') dV' + \int_{V}^{V^{(1)}} \max \left[ 0, \tilde{V} - V \right] dF(V') 
\]

(34)

\[
w \geq \underline{w} 
\]

(35)

\[
U \leq Y \leq \tilde{V} \right\}.
\]

(36)

For this to be a proper representation of the original optimal contracting problem, the promise keeping constraint must be binding for all continuation utility promises in the problem, that is, for any optimal choice of \( Y(V) \) it must be that \( \nu(Y(V)) > 0 \). Indeed, this
will be the case. The only time, the promise keeping constraint may not be binding is at
the very outset of the contract. Here, it is possible that the firm may immediately want to
jump the promise value up to $V$ after which the contract will proceed with a binding utility
promise constraint ever after.

Therefore, consider the initial point of the contract. One obtains the first order conditions
where $\eta(V)$ is the Lagrange multiplier on the minimum wage constraint,

$$-1 + \nu(V) u'(w(V)) + \eta(V) = 0$$

$$\Pi'(V) = -\nu(V)$$

$$\mu(Y(V)) = \frac{-\beta (1 - d) f_{Y(V)}'}{\hat{F}'(s(Y(V)))}$$

$$\nu(V) + \Pi'(Y(V)) = -\Psi(Y(V))$$

$$\Psi(Y(V)) = \frac{\mu(Y(V)) \hat{F}(Y(V))}{1 - s(Y(V)) \hat{F}(Y(V))}.$$

Define $\underline{V}$ as the utility promise that maximizes the firm’s profit function,

$$\underline{V} = \arg \max_V \Pi(V).$$

Assuming that the profit function is differentiable at $\underline{V}$, the first order condition requires,

$$\Pi'(\underline{V}) = 0.$$ 

Therefore, it must be that the utility promise constraint is not binding at $\underline{V}$, that is $\nu(\underline{V}) = 0$. This in turn means that the lower wage bound must be binding, $\eta(\underline{V}) = 1$ and $w(\underline{V}) = w$. Notice that absent a lower wage bound and given assumption A2, $\underline{V}$ does not exist (in this case the profit function is everywhere monotonically decreasing in $V$). It must be that for any $V < \underline{V}$, the utility promise constraint is slack, $\nu(V) = 0$. As proof assume to the contrary: Take any $V < \underline{V}$ and assume that $\nu(V) > 0$. But this immediately implies that $\Pi'(V) = -\nu(V) < 0$, which implies that $\underline{V} < V$, which is a contradiction.

The continuation utility promise at $\underline{V}$ is the solution to the equation,

$$\Pi'(Y(\underline{V})) = -\Psi(Y(\underline{V})) < 0,$$

which means that the continuation utility will put the problem into the strictly decreasing part of the profit function where the utility promise constraint is binding and the analysis
in the previous section applies from that point on. Specifically, the utility promise profile is increasing in contract duration. Therefore, the recursive representation of the problem is a proper representation of the firm’s wage mechanism design problem.

The search intensity is directly determined by the continuation utility choice,

\[ s(V) = c'(1-d) \left( \int_{Y(V)}^{\hat{V}} \hat{F}(V') dV' + \int_{\hat{V}}^{V} \max \left[ 0, \tilde{V}(V') - \bar{V} \right] dF(V') \right). \]

Therefore, the \( V \) can also be written as,

\[ V = u(w) + S(Y(V)) + \beta \left[ dU + (1-d) Y(V) \right], \tag{37} \]

where,

\[ S(Y) = \max_s \left[ -\hat{c}(s) + \beta (1-d) s \left[ \int_{Y}^{\hat{V}} \hat{F}(V) dV + \int_{\hat{V}}^{V(1)} \max \left[ 0, \tilde{V}(V) - \bar{V} \right] dF(V) \right] \right]. \]

Notice, an important aspect of the behavior of the contract when the lower wage bound is binding. Consider the contract for the utility promise that happens to unconstrained pick a starting wage exactly equal to the minimum wage. The continuation utility promise is given by,

\[ \Pi'(Y(V)) = -\frac{1}{u'(w)} - \Psi(Y(V)) < -\Psi(Y(V)), \]

which means that the continuation utility for this contract is greater than \( Y(V(p)) \). Both contracts have starting wage \( w \), but the latter grows the wage slower.

Denote by \( \tilde{V}(p) \) the utility promise such that the firm exactly chooses to offer the worker the minimum wage, but is not constrained by it, that is \( \eta(V(p)) = 0 \) and \( w(V(p), p) = w \).

By the modified Assumption A2, it must be that \( \Pi'(\tilde{V}(p), p) = -1/u'(w) < 0 \). Therefore, there exists a range of utility promises \( V \in [V(p), \tilde{V}(p)] \) where the minimum wage constraint is binding. In this range, both the minimum wage bound and the utility promise constraint are binding, \( \nu(V) > 0 \) and \( \eta(V) > 0 \), and the contract is trivially determined by the binding constraints:

\[ w(V) = w \]
\[ V = u(w) + S(Y(V)) + \beta \left[ dU + (1-d) Y(V) \right] \]
\[ s(V) = c'^{-1} \left( \beta (1-d) \left[ \int_{Y(V)}^{\hat{V}} \hat{F}(V') dV' + \int_{\hat{V}}^{V} \max \left[ 0, \tilde{V}(V') - \bar{V} \right] dF(V') \right] \right). \]
Take any two utility promises \( (V^0, V^1) \in \left[ V(p), \tilde{V}(p) \right] \) such that \( V^0 < V^1 \). Both contracts start with the same wage \( w(V^0) = w(V^1) = \overline{w} \). However, the lower utility promise contract grow utility more slowly, \( Y(V^0) < Y(V^1) \). Furthermore, search intensity is higher for the lower utility promise \( s(V^0) > s(V^1) \). It is of course perfectly possible to have that both \( (V, Y(V)) \in \left[ V(p), \tilde{V}(p) \right] \), meaning that the wage contract can be initially flat at the lower wage bound.

Notice that there is a meaningful distinction between whether a lower wage bound is a result of a violation of assumption A2 or as a result of a constraint like \( w \geq \overline{w} \). In the case where \( \lim_{w \to \overline{w}} u'(w) = \infty \), the interval \( \left[ V(p), \tilde{V}(p) \right] \) is empty. In a continuous time version of the model, the implication would be that the minimum wage would not be mass point in the wage distribution. In the case where \( u'(w) \) is finite, the equilibrium wage distribution can have a mass point at the minimum wage.

### 3.1.1 Numerical example continued

Take the numerical example in section 2.8 and add the constraint \( w \geq 0 \). Figure 5 shows a few key contract objects across firm types. The right panel shows the willingness to pay, \( \overline{V}(p) \), the minimum utility offered, \( V(p) \), and the minimum utility promise where the wage constraint remains non-binding. It is seen that \( V(p) \) is increasing in firm type to the point where workers in firms with productivity \( p \leq 0.25 \) will expect positive rent extraction gains from third party meetings.
Figure 6: Median Contract

Note: The median firm has $p_{med} = 0.39$. The key utility promise objects are, $\bar{V}(p_{med}) = 1.855, \tilde{V}(p_{med}) = 1.877, \check{V}(p_{med}) = 1.952$.

The right panel shows the impact of positive ex ante rent extraction on search intensities. $\lambda(\bar{V}(p), p)$ shows the search intensity of a worker just hired from either unemployment or from a firm with willingness to pay $\bar{V}(p') \leq \bar{V}(p)$. It is seen that this search intensity is decreasing in firm type. This is in contrast to the no ex ante rent extraction case in section 2.5, where $\lambda(U, p)$ is increasing in firm type. In the unconstrained case, $\lambda(U, p)$ is monotonically increasing in $p$ because the gains to search $\int_U^{\bar{V}(p)} F'(V') dV'$ are monotonically increasing in $\bar{V}(p)$. In the constrained case, two forces combine to flip the relationship: The minimum utility promise $\check{V}(p)$ is increasing in $p$ fast enough to reduce the gains from rent extraction from the current firm, and second, the expected gains from third party rent extraction are also decreasing in firm type. The latter incentive is shown directly in the search intensity at full rent extraction, $\lambda(\tilde{V}(p), p)$. At full rent extraction with the current
firm, any search is purely motivated by possible rent extraction from outside firms. Workers in firms with productivity $p > 0.25$ have no incentive to search since $\bar{V}(p) > V(1)$, that is, their current utility exceeds that of the minimum utility promise of even the most productive firm. But workers in matches less productive than $p = 0.25$ have strictly positive returns to search.

The added incentives to search at the lower end of the firm hierarchy shows up in a reduced match survival rate at the lower end of the firm hierarchy. In the example, the search intensity of the unemployed worker coincides with $\lambda(\bar{V}(0), 0)$ which is the highest search intensity in the economy. So, the example now provides a likely attractive feature that the hardest searching worker in the economy is the unemployed worker.

In Figure 6 I illustrate the optimal contract for the median firm over the utility support for the contract. The profit function has zero derivative at the lower support bound, $V(p_{med})$, and is monotonically decreasing over the support. The contract search intensity is also monotonically decreasing in the utility promise. The lower left panel shows the wage function, where the minimum wage is binding over the range $[V(p_{med}), \bar{V}(p_{med})]$. In this range, the utility promise growth rate is increasing the promise. In the utility promise range where the minimum wage constraint is not binding, the wage is strictly increasing in the utility promise and the utility promise growth rate is strictly decreasing in $V$. Thus, a contract that starts with a utility promise of $V = V(p_{med})$ will specify a minimum wage rate for roughly the first 3 months after which it will start increasing monotonically in duration.

I illustrate the workings of the contracts by considering the evolution of an employment relationship in the median firm type with a worker hired directly out of unemployment. Interestingly, the evolution of the median employer contract does not look radically different from the unconstrained case in Figure 3. Of course, the monthly earnings, as shown in the middle right panel, now start a zero instead of negative earnings, where the first two monthly wages are at the minimum wage, but the earnings path otherwise come to look very similar to the unconstrained case. The monthly separation hazard is reduced somewhat in the first few months but subsequently coincide quite closely with the unconstrained case. The median firm’s willingness to pay exceeds $V(1)$. Hence, its workers have no expectation of third party rent extraction same as in the unconstrained case. Hence, the lower wage bound affects the contract only directly from the lower end through the firm’s choice of
Note: The contract is shown from the end of the first month and forward. The first observation for the monthly earnings are the accumulated earnings during the first month. The solid black lines show the evolution of the contract based on the utility promise evolution in equation (29) which assumes no outside offers that may trigger renegotiation. The green lines are based on the utility promise evolving according to equation (30), which is the case where outside offers may trigger renegotiation of the contract.

$V(p)$ and the evolution of wages for the lower end of utility promises. In lower rank firms, workers face ex ante rent extraction incentives and search is now more intensive, and the contracts will show more of an impact from the constrained environment.
4 Socially Efficient Search

The Mortensen rule, Mortensen (1982), of efficient search dictates that the initiator or “match maker” of a match also receive all the surplus from it. In the partial setting that I have worked with so far, workers are fully responsible for the creation of matches, yet in the no ex ante rent extraction version of the model all the rents associated with unemployed search go to the firms. Clearly, workers in this state must search too little from a social point of view. It is actually quite bad: If the model does not provide the unemployed worker with a costless strictly positive base level of offer arrivals, unemployment becomes a trap and the steady state economy collapses to zero employment.

At the other extreme, consider a worker currently employed with the most productive firm. Any search by this worker is socially inefficient since it cannot result in the creation of a more productive match. Yet, as long as the worker is receiving less than all of the rents from the match, he or she will engage in a strictly positive level of search, which is socially too much. In fact, the most intensely searching worker in the economy absent any ex ante rent extraction is the worker who has just been hired out of unemployment into the most productive firm.

Thus, it should be clear that the economy will not be socially efficient. The direct application of the Mortensen rule is to give the worker the entire production flow from a match. Constrain the social planner to not be able to smooth consumption across individuals and states. For simplicity, consider the continuous time version of the model. Let the social welfare criterion for a given state of the economy, \( G(p) \), be given by,

\[
W = \max \int_0^1 V(p) dG(p),
\]

where

\[
V(p) = \max_{s(t)} E \left[ \int_0^\infty e^{-rt} [u(p) - c(\lambda(t))] \right]
\]

is the discounted stream of future utilities and \( \lambda(t) \) is the choice of state conditional search

\footnote{This is also true for the search by workers who are extracting all the rents from their current relationships.}
intensities. The functional equation for this expression is,

\[(r + \delta) V(p) = \max_\lambda \left[ u(p) - c(\lambda) + \delta V(0) + \lambda \int_p^1 [V(p') - V(p)] d\Phi(p') \right] \]

\[= \max_\lambda \left[ u(p) - c(\lambda) + \delta V(0) + \lambda \int_p^1 V'(p') \hat{\Phi}(p') dp' \right] \]

\[= \max_\lambda \left[ u(p) - c(\lambda) + \delta V(0) + \lambda \int_p^1 \frac{u'(p') \hat{\Phi}(p')}{r + \delta + \hat{\Phi}(p') \lambda(p')} dp' \right],\]

where the second line follows from integration by parts and the third by the envelope condition. Following the notation scheme, \(\hat{\Phi}(p) = 1 - \Phi(p)\). Therefore, the social planner will for each worker dictate a state dependent search intensity characterized by the first order conditions,

\[c'(\lambda^*(p)) = \int_p^1 \frac{u'(p') \hat{\Phi}(p')}{r + \delta + \hat{\Phi}(p') \lambda^*(p')} dp',\]

\(\lambda^*(p)\) is the constrained socially optimal level of search. Obviously, it is not tenure dependent. It is monotonically decreasing in the type of the firm.

If on the other hand, firms spend resources as part of the match creation process, then the Mortensen rule dictates that for search to be efficient, firms must receive some of the ex post rents from matches. As in the case of the Hosios rule, the setting calls for a specific balance in rent extraction. One can imagine labor market policies and institutions, such as for example the minimum wage as playing a possibly important role in this case.

## 5 Steady state

Denote by \(u\) the mass of workers that are unemployed. Similarly, let \(e = 1 - u\) denote the mass of employed workers. In addition, denote by \(g(V,p)\) the mass of workers that are employed in productivity \(p\) firms with utility promise \(V\). Denote by \(G(V,p)\) the associated cumulative distribution function. Denote by \(\lambda^0\) the search intensity of an unemployed worker. In steady state, worker inflows and outflows must equal each other, which implies,

\[u\lambda_0 = e\delta = (1 - u)\delta.\]

Define the firm type threshold \(\underline{p}(V)\) as the highest firm type such that \(\underline{V}(p) \leq V\),

\[\underline{p}(V) = \sup \{p \in [0, 1]|\underline{V}(p) \leq V\}.\]
As long as $\bar{V}(p)$ is monotonically increasing, $\bar{V}(p) \leq V$ for any $p \leq \underline{p}(V)$. The steady state match distribution is characterized by the same type of equality between inflows and outflows which implies,

$$u\lambda_0 \Phi \left( \min \left( p, \underline{p}(V) \right) \right) = e \left\{ \hat{F}(V) \int_0^p \int_U^{\min[V_0(p'),V]} \lambda(V',p') dG(V',p') \\
+ \delta G(V,p) + \int_{\bar{V}_{0}^{-1}(V)}^{p} \hat{V}(V,p') g(V,p') dp' \right\}$$

$$\delta \Phi \left( \min \left( p, \underline{p}(V) \right) \right) = \int_0^p \int_{U}^{\min[V_0(p'),V]} \left[ \delta + \hat{F}(V) \lambda(V',p') \right] dG(V',p') \\
+ \int_{\bar{V}_{0}^{-1}(V)}^{p} \hat{V}(V,p') g(V,p') dp'.$$

It is worth noting that keeping track of the distribution of utility promises in the match distribution is quite straightforward in the continuation time formulation. The above expressions equalize the flows in and out of the pools of experience $i$ workers in productivity $p$ or less firms that are in receipt of utility $V$ or less, $e_iG_i(p,V)$. The change in utility promise over time in the contracts results in an outflow equal to, $\int_{\bar{V}_{0}^{-1}(V)}^{p} \hat{V}(V,p') g_0(V,p') dp'$. Given that utility promises change smoothly over time in the optimal contracts, the only workers that can leave the pool at a given point in time, are the ones in the pool that receive exactly $V$, which can happen in a range of different productivity $p' \in \left[ \bar{V}_{0}^{-1}(V), p \right]$ firms. These workers leave the pool at rate, $\dot{V}_0(V,p')$.

### 5.1 Frictional wage dispersion

In this section I calibrate the model to the favored settings in Hornstein et al. (2011) (henceforth HKV) and show that even in the absence of worker heterogeneity, the model can produce levels of wage dispersion as seen in the data.\textsuperscript{8} It is an important result because

\textsuperscript{8}A superficial reading of HKV might produce the impression that frictional labor market models cannot produce substantial wage dispersion absent worker heterogeneity. Thus, the results in this section would seem to be in stark contrast to such a message. However, HKV do suggest in the latter sections in their paper that models related to the one in this paper are quite capable in this respect. On-the-job search being the key ingredient that resolves the issue of why unemployed workers would so readily accept seemingly any offer of employment while facing a labor market with significant dispersion in wage outcomes. Devine and Kiefer (1991) argue a related point stating that systematic unemployment duration differences across workers are best understood as a result of variation in offer arrival rates rather than as an offer rejection phenomenon.
inclusion of productive worker heterogeneity in the model is likely part of empirical studies of wage dynamics in this setting. The result states that separate identification of the distribution of worker heterogeneity in such an estimation cannot by itself be done based on fitting cross-section wage dispersion moments since they can be fully explained even in the complete absence of worker heterogeneity.

In the calibration I use the lower wage bound as the only mechanism that generates ex ante rent extraction which is essential to give unemployed workers incentive to search. I am matching the model up against HKV’s favored measure of wage dispersion, the mean-min (Mm) ratio. Hence, the minimum wage is performing double duty. Nevertheless, the model has enough flexibility to match the facts with quite a few degrees of freedom to spare. A more negative implication is that the key measures in HKV do not provide strong identification of even the productive heterogeneity of firms in the model.

The utility function is specified to be CRRA, \( u(c) = c^{1-\theta}/(1-\theta) \). The search cost function remains specified as a power function according to equation (28). As in the previous examples, assume \( \Phi(z) \) is a beta distribution with parameters \((\beta_0, \beta_1)\) over \([0,1]\). Instead of normalizing the lowest productivity at zero, let the firm productivity index, \( p \) map into revenue according to a linear production function \( f(p) = \alpha_0 + \alpha_1 p \). Furthermore, the unemployed income is normalized at \( b = 1 \). With this, the model has the following parameters \((r, \delta, \theta, \alpha_0, \alpha_1, w, c_0, c_1, \beta_0, \beta_1)\). I calibrate to the US economy following HKV. I set the interest rate at a 5% annual rate, \( r = 0.05 \), and the utility function is set at its conventional level of \( \theta = 2 \). Following Shimer (2012) the monthly employment to unemployment rate is set at 2% (prime age men, Figure 3), which translates to \( \delta = 0.24 \).

HKV focus on 3 particular measures in their discussion of search models: (1) The Mean-min wage ratio, which is the ratio of the average wage in the economy relative to the minimum wage, and (2) the value of non-market time relative to value of market time, specifically the ratio of unemployment benefits to the average earned wage, and (3) the duration of unemployed search.

HKV report mean-min values for the U.S. labor market between 1.7 and 1.9. This is the 50-10 percent ratio, so since in the model I will use the values of the mean and min wages, I will match the model up against a Mm ratio of 2. In Shimer (2005) the \( b/E[w] \) fraction is estimated at 0.4. HKV present arguments in favor of both larger and smaller values. I will
match to the 0.4 ratio as the main fact to match, but I will show that the model can match much larger values as well, which for the purpose of demonstrating the model’s ability to generate frictional wage dispersion is the stricter test.

The duration of unemployed search is in HKV set at 2.3 months which translates to an annual job finding rate of 5.2. This is likely an overly optimistic estimate of the offer arrival rate associated with job search in the U.S. For example, Fujita and Moscarini (2014) show that about half of all unemployment exits are recalls. Disregarding recalls, Fujita and Moscarini (2014) estimate that the monthly probability of finding a new job is 0.147, which translates to an average unemployment duration of 6.3 months and an annual job finding rate in my model of 1.9.

In the core calibration, I will calibrate up against the 5.2 unemployed job finding rate because it represents the stricter test in terms of making the model generate wage dispersion. I then include a calibration with an annual job finding rate of 1.9 as the likely better benchmark for the purpose of matching unemployed search for new job opportunities in the United States. But eliminating recalls from the UE rate does imply that they should also be eliminated from the EU rate. The unemployment rate associated with a job finding rate of 5.2 and a job destruction rate of 0.24 is $u = 0.044$. If I require that unemployment stay the same, then a reduction of the job finding rate to 1.9 implies a layoff rate of $\delta = 0.0876$.

With 7 free parameters ($\alpha_0, \alpha_1, w, c_0, c_1, \beta_0, \beta_1$) and only 3 moments to match, I am under identified. I set search cost to be quadratic, $c_1 = 1$ and set the firm heterogeneity distribution at $(\beta_0, \beta_1) = (2, 3)$. The pdf is shown in Figure 1. Furthermore, I set $\alpha_0 = \max[b, \bar{w}]$.

In Table 2, I summarize the model calibration for different scenarios. Regardless of scenario, the model successfully fits the calibration moments. Scenario 1 represents the core calibration in HKV. Given the normalization of $b = 1$, the requirement that the mean-min ratio be 2 and the ratio of unemployment benefits to the average wage be 0.4, implies that the minimum wage must be $w = 1.25$. It is seen that the model fits the calibration moments with productivity distribution support where the most productive firm is about 3 times as productive as the least productive firm. To the extent that these differences show up proportionally in observed labor productivity statistics, this is likely a somewhat conservative setting for firm productivity dispersion. For example, in Lentz and Mortensen (2008), the 95-5 percentile in the labor productivity distribution in Denmark is around 4.
In scenario 2, I lower the job finding rate to 1.9, which the model fits by increasing search cost. By itself, the increased search cost, will reduce the amount of wage growth in the economy, and so to maintain the mean-min ratio of 2, the calibration increases the firm productivity dispersion support upward. In scenario 3, I combine the lower job finding rate with a lower layoff rate so as to maintain the same level of unemployment as in scenario 1. In this case search cost remain roughly the same as in scenario 2, but the firm productivity distribution reverts back to almost that of scenario 1.

In scenario 4, relative to scenario 1 I increase the $b/E[w]$ ratio to 0.75. Given the normalization of $b = 1$, this implies a lower wage bound of $w = 2/3$. And finally in scenario 5, which one could consider the Hagedorn and Manovskii (2005) extreme, I take the $b/E[w]$ ratio all the way to 0.95, which in this model is a rather extreme position. The model fits the increasing ratio of benefits to average wages by lowering search costs and by narrowing the firm productivity support.

Scenario 5 is a fairly extreme calibration. Since $b$ bounds the productivity support below, requiring that $b$ is 95% of the average wage realization puts a tight constraint on the how much upward wage dispersion there can be in the equilibrium, at least for the given beta distribution parameters. For the given productivity distribution shape, there is very little productive heterogeneity in the equilibrium. Certainly, the calibration would perform poorly in terms of measures of productivity dispersion across firms if confronted with such data. However, the mean-min ratio does not register this reality and by this measure, there is as much wage dispersion in scenario 5 as in scenario 1.

Finally, scenario 6 takes the core calibration and demonstrates that the model can readily produce much larger measures of wage dispersion should it be so asked. Specifically, the model is calibrated to produce a mean-min ratio of 10, while otherwise fitting the same

### Table 2: Model Calibrations

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$c_0$</th>
<th>$w$</th>
<th>$\alpha_0$</th>
<th>$\alpha_1$</th>
<th>$\delta$</th>
<th>$E[w]/w$</th>
<th>$b/E[w]$</th>
<th>EU rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.184</td>
<td>1.25</td>
<td>1.25</td>
<td>2.57</td>
<td>0.240</td>
<td>2.00</td>
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<td>5.20</td>
</tr>
<tr>
<td>2</td>
<td>0.450</td>
<td>1.25</td>
<td>1.25</td>
<td>3.24</td>
<td>0.240</td>
<td>2.00</td>
<td>0.40</td>
<td>1.90</td>
</tr>
<tr>
<td>3</td>
<td>0.490</td>
<td>1.25</td>
<td>1.25</td>
<td>2.61</td>
<td>0.088</td>
<td>2.00</td>
<td>0.40</td>
<td>1.90</td>
</tr>
<tr>
<td>4</td>
<td>0.110</td>
<td>0.67</td>
<td>1.00</td>
<td>0.67</td>
<td>0.240</td>
<td>2.00</td>
<td>0.75</td>
<td>5.20</td>
</tr>
<tr>
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<td>0.045</td>
<td>0.53</td>
<td>1.00</td>
<td>0.12</td>
<td>0.240</td>
<td>2.00</td>
<td>0.95</td>
<td>5.20</td>
</tr>
<tr>
<td>6</td>
<td>0.251</td>
<td>4.00</td>
<td>4.00</td>
<td>104.07</td>
<td>0.240</td>
<td>10.00</td>
<td>0.40</td>
<td>5.20</td>
</tr>
</tbody>
</table>
moments as in scenario 1. The model does so by increasing productivity dispersion.

6 Concluding remarks

The paper studies wage dynamics in an on-the-job search model with hidden search. Firms have incentive to backload wage contracts so as to preempt workers from engaging in search that is primarily motivated by rent extraction from the current match. Wages are increasing within jobs in tenure and in outside offer arrivals. Also within a job, search intensity is decreasing in tenure and outside offer arrivals. Unlike the standard on-the-job search model, wages can decrease between jobs and search intensity can increase, although neither object has to. Lifetime utility is monotonically increasing within an employment spell, both within and between jobs.

An employment contract specifies a wage and incentive compatible search intensity profile as well as an outside offer response function. Hence, the setup admits both wage posting and offer matching as special cases. The analysis requires contracts to be renegotiation proof, which implies that optimal employment contracts match outside offers as in Postel-Vinay and Robin (2002).

The analysis has established existence and uniqueness of the core mechanism as well as characterization of the optimal contract. In addition, design constraints such as minimum wages have been shown to have a profound impact on rent division properties of the model.

Hidden search induces firms to design contracts that deliver wage growth based on the worker’s implicit threat that absent such a promise, she will force the firm’s hand by summoning outside offers. Thus, wage growth in the model is a result both of actual outside offers as well as the threat of them. The latter force turns out to be quite powerful. In particular, it is possible to generate very fast wage growth in the model so as to effectively overturn the likely negative feature in Postel-Vinay and Robin (2002) that the lowest wage in the economy is paid by the most productive firm. In the numerical examples in the paper, the more productive firms pay more even at high frequency measurement such as monthly wages than the less productive firms even to workers just hired out of unemployment.

That wage growth is tied to the threat of search and not necessarily to the actual arrival of outside offers also severs a tight link between observed job-to-job transitions and wage
growth in this type of model. Seen in a positive light, it adds flexibility to the model. However, from an identification point of view, one must now seek identification strategies for the worker’s search cost function to properly understand wage dynamics.

The model produces rich wage dispersion and dynamics. This is in part demonstrated by its ability to match the desired wage dispersion measures in Hornstein et al. (2011) without introducing worker heterogeneity into the model.
A Solution Algorithm for Continuous Time Model.

A.1 The optimal contract with possibly binding minimum wages.

In the continuous case, there is no contraction mapping to work with for a value function iteration scheme. Of course, one can iterate on equation (10) for very small $\Delta$, however one would expect very slow convergence. Instead, one can adopt the following algorithm. The algorithm seeks to solve the following problem:

$$[r + \delta] \Pi (V) = \max_{(w, \lambda, V) \in \Gamma (V)} \left[ p - w + \lambda \int_{V}^{\bar{V}(p)} \Pi' (V') \hat{F} (V') dV' + \Pi' (V) \hat{V} \right],$$

where the set of feasible choices $\Gamma (V)$ is given by,

$$\Gamma (V) = \left\{ (w, \hat{V}, \lambda) \in \mathbb{R}^3 \left| u (w) - c (\lambda) + \delta U + \lambda \left[ \int_{V}^{\bar{V}(p)} \hat{F} (V') dV' + \int_{V}^{\bar{V}(1)} \max \left[ 0, \bar{V} (V') - \hat{V} \right] dF (V') \right] + \hat{V} = (r + \delta) V \right\}. \quad (38)$$

First order conditions are given by,

$$-1 + \nu (V) u' (w) + \eta (V) = 0$$

$$\Pi' (V) + \nu (V) = 0$$

$$\int_{V}^{\bar{V}(p)} \Pi' (V') \hat{F} (V') dV' + \mu (V) c'' (\lambda) = 0$$

$$\Pi'' (V) \hat{V} (V) = - \mu (V) \hat{F} (V)$$

$$(r + \delta) V = u (w (V, p)) + S (V, p) + \delta U + \hat{V} (V, p),$$

The first two equations imply,

$$\Pi' (V) = \frac{\eta (V) - 1}{u' (w)}.$$

The solution algorithm is a fixed point search in $\bar{V} (p)$. Make some initial guess of this object. The following algorithm then maps out an implied value, $\hat{V} (p) = TV (p)$. The model is solved upon finding the fixed point of the mapping.
If the lower wage bound is not binding, then it must be that \( \nu(V) > 0 \), that is the promise keeping constraint is binding, and the slope of the profit function is directly linked to the wage level by, \( \Pi'(V, p) = -1/u'(w(V, p)) < 0 \). If the minimum wage constraint is binding, \( \eta(V) > 0 \), and the worker’s offered utility is defined by,

\[
(r + \delta) V = u(w) + S(V, p) + \delta U + \dot{V}(V, p),
\]

where

\[
S(V, p) = \max_{\lambda} \left[ -c(\lambda) + \lambda \left[ \int_V^{\bar{V}(p)} \hat{F}(V') dV' + \int_{\bar{V}(p)}^{\bar{V}(1)} \max \left[ 0, \bar{V}(V') - \bar{V} \right] dF(V') \right] \right].
\]

There is a natural conjecture to a threshold solution here: Define \( \tilde{V}(p) \) such that,

\[
\eta(V, p) = 0 \forall V \geq \tilde{V}(p)
\]

\[
\eta(V, p) > 0 \forall V < \tilde{V}(p).
\]

The minimum utility offered by a firm is in this context defined by the utility promise such that,

\[
\Pi'(\bar{V}(p), p) = 0
\]

\[\Uparrow\]

\[
\eta(\bar{V}(p), p) = 1.
\]

Hence, for utility promises in the range \( \left[ \frac{u(0)}{r}, \bar{V}(p) \right] \), there are two important thresholds \( \bar{V}(p) \leq \tilde{V}(p) \), such that profits are increasing for \( V < \bar{V}(p) \), and a firm would voluntarily increase the offered utility over \( V \) all the way up to \( \bar{V}(p) \). For \( \bar{V}(p) \geq V \geq \bar{V}(p) \) profits are decreasing in the utility promise, the wage is at the lower bound and there is a positive growth rate in the utility promise. For \( V \geq \bar{V}(p) \), the lower wage bound is not binding and profit function is downward sloping.

The algorithm below describes how to solve for the optimal contract for utility promises in the support \( \left[ \tilde{V}(p), \bar{V}(p) \right] \). The gradual descending solution from the willingness to pay eventually reveals \( \tilde{V}(p) \), as the lowest utility promise such that the optimally chosen wage is exactly at the lower wage bound, but not bound by it. The solution algorithm can then proceed in a descending manner from \( \tilde{V}(p) \) toward \( \bar{V}(p) \) using the following insights: Since
the lower wage bound is binding, the utility promise growth rate must satisfy,

\[
\dot{V}(V, p) = (r + \delta) V - u(w) - S(V, p) - \delta U.
\]

At least for \( V \) close enough to \( \tilde{V}(p) \) this will be a positive expression. Thus, the contract is directly dictated by the binding constraints (lower wage bound, worker lifetime utility definition, and the incentive compatibility). Therefore, firm profits evolve according to the first order differential equation,

\[
\Pi'(V, p) = \frac{[r + \delta] \Pi(V, p) - p + w - \lambda(V, p) \int_{V}^{\tilde{V}(p)} \Pi'(V') \hat{F}(V') dV'}{\dot{V}(V, p)},
\]

which is initialized at \( V = \tilde{V}(p) \). At this point, it is known that \( \Pi'\left(\tilde{V}(p), p\right) = -1/u'(w) \) and the expressions on the right hand side are also all known from the solution algorithm that determined \( \tilde{V}(p) \) in the first place. By the envelope theorem, the second derivative of the profit function is also known,

\[
\Pi''(V) = \frac{\hat{F}(V) \int_{V}^{\tilde{V}(p)} \Pi'(V') \hat{F}(V') dV'}{\dot{V}(V, p) \eta'(\lambda(V, p))}.
\]

Thus, one can shoot out from the initialization point \( \tilde{V}(p) \) and downwards. By the differential equation, one can then map out \( \Pi(V), \Pi'(V), \) and \( \Pi''(V) \). This then also implies that the wage bound multiplier can be determined by, \( \eta(V, p) = 1 + u'(w)\Pi'(V, p) \). The downward projection will eventually produce a profit function derivative of \( \Pi'(V, p) = 0 \), which then provides the lower threshold, \( \tilde{V}(p) \). It is natural to stop the solution algorithm at this point, since any utility promise \( V < \tilde{V}(p) \) is not renegotiation proof.

For the part of the problem where \( V \in \left[\tilde{V}(p), \bar{V}(p)\right] \) adopt the following algorithm: First, determine \( \tilde{V}(p) \) and \( U \). With this, the optimal contract for a given type can be solved for a given firm type independent of the other firm types (which also suggests an obvious parallelization strategy if multiple processors are available). For any given firm type, the
contracting problem is initialized at the willingness to pay, $\bar{V}(p)$. Here it is known that,

\[
\begin{align*}
\dot{V}(\bar{V}(p), p) &= 0 \\
w(\bar{V}(p), p) &= p(p) \\
\lambda(\bar{V}(p), p) &= 0 \\
\Pi(\bar{V}(p), p) &= 0 \\
\Pi'(\bar{V}(p), p) &= \frac{-1}{u'(p(p))}.
\end{align*}
\]

Discretize the state space \(\{V_i\}^N_{i=1}\) such that \(V_1 = U\) and \(V_N = \bar{V}(p)\). There will be more curvature in the problem as \(V\) goes to \(\bar{V}(p)\) and so it may make sense to adopt a grid that is denser on the right hand side, although this is not essential. Equation (21) immediately provides the search choice for any utility promise grid point \(\{\lambda_i\}^N_{i=1}\). It then remains to determine \(\{w_i, \hat{V}_i, \Pi_i, \Pi'_i\}^N_{i=1}\). Define a distance function \(D_i(w) = (V_i - \hat{V}_i(w))^2\), where \(\hat{V}_i(w)\) is obtained as follows,

1. Set \(w_{1i} = w\).

2. By equation (20), this provides \(\Pi'_i = -1/u'(w_i)\).

3. Equation (20) also implies that \(\Pi''(V) = u''(w(V)) w'(V)/u'(w(V))^2\). Approximate \(w_i' \approx (w_{i+1} - w_i)/(V_{i+1} - V_i)\), which then provides \(\Pi''_i = u''(w_i) w_i'/u'(w_i)^2\).

4. Equation (22) now provides \(\hat{V}_i = \hat{F}(\hat{V}_i) \int_{V_i}^{V_i(p)} \Pi'(V') \hat{F}(V') dV' / [c''(\lambda_i) \Pi''_i]\). The integral evaluation is done by interpolation over the points \(\{\Pi'_i\}^N_{i=i}\).

5. By equation (24), this finally delivers the implied utility promise for the given wage guess, \((\rho + \delta) \hat{V}_i = u(w_i) - c(\lambda_i) + \delta U + \lambda_i \int_{V_i}^{V_i(p)} \hat{F}(V') dV' + \hat{V}_i\).

The optimal wage contract is found such that \(D(w_i) = 0\) for all \(i = 1, \ldots, N\). Start the algorithm from \(i = N - 1\) and loop down to \(i = 1\).
References


