

Regression Discontinuity Design with Many Thresholds*

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Abstract

In recent years, numerous studies have employed regression discontinuity designs with many cutoffs assigning individuals to heterogeneous treatments. A common practice is to normalize all of the cutoffs to zero and estimate only one effect. This procedure identifies the average of local treatment effects weighted by the observed relative density of individuals at the existing cutoffs. However, researchers often want to make inferences on more meaningful average treatment effects (ATE) computed over general counterfactual distributions of individuals rather than simply the observed distribution of individuals local to existing cutoffs. In this paper, we propose a root- n consistent and asymptotically normal estimator for such ATEs when heterogeneity follows a non-parametric smooth function of cutoff characteristics. In the case of parametric heterogeneity, observations are optimally combined to minimize the mean squared error of the ATE estimator. Inference results are also provided for the fuzzy regression discontinuity case, where the parametric heterogeneity assumption yields identification of treatment effects on individuals who comply with at least one of the multiple treatments.

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1 Introduction

One of the fundamental problems in economic analyses with observational data is that we do not see the counterfactual scenario needed to make causal inferences. When the researcher has a theoretical relationship among variables in mind, it is the task of econometrics to derive minimal conditions for this relationship such that the causal effect is identified and feasible to estimate (White and Chalak (2013), Heckman and Vytlačil (2007)). Applications of regression discontinuity design (RDD) have become increasingly popular in economics since the late 1990s (Black (1999), Angrist and Pischke (1999), Van der Klaauw (2002)). One of RDD's main advantages is the identification of a local causal effect under minimal functional form assumptions. More recently, with the increasing availability of richer data sets, there have been many applications with multiple cutoffs and treatments (e.g. Ajayi (2014), Hastings, Neilson, and Zimmerman (2013), Pop-Eleches and Urquiola (2013), Garibaldi, Giavazzi, Ichino, and Rettore (2012), Saavedra (2008)). Existing one-cutoff RDD methods can be applied to each cutoff individually but yield local effects that are estimated using only a few observations near each cutoff. Researchers often prefer one takeaway summary effect that can be more precisely estimated by using all the data. The ability to combine observations from various cutoffs depends crucially on heterogeneity assumptions connecting local treatment effects.

Many existing applied studies with multiple cutoffs simply normalize all cutoffs to zero and use the one-cutoff estimator. If treatment effects are heterogeneous, this normalization procedure estimates an average of local treatment effects weighted by the relative density of individuals near each of the cutoffs (Cattaneo, Keele, Titiunik, and Vazquez-Bare (2014)). Such an average effect would be a meaningful summary measure only in two cases: (i) all local treatment effects are identical and the weighting scheme does not matter; (ii) the researcher is interested in the average effect of heterogeneous treatments only on the observed distribution of individuals near the existing cutoffs of such treatments. However, researchers are often interested in combining observed data with assumptions weaker than (i) to make

inferences on counterfactual scenarios more general than (ii).¹

The inference procedure of our paper estimates an average of local treatment effects (ALTE) that is a more valuable summary measure than the average effect identified by the normalization procedure because the weighting scheme can be explicitly chosen according to the counterfactual scenario of interest. For applications with substantial variation in cutoff values, we also propose a corrected weighting scheme that allows for inference on an even more valuable summary measure, which is the average treatment effect (ATE) computed over an explicitly chosen distribution of individuals including individuals away from existing cutoffs.

We motivate our framework for RDD with many thresholds using a simple example based on Pop-Eleches and Urquiola (2013), PEU from now on. They study the assignment of students to more or less elite high schools based on test scores where every town has its own set of minimum admission cutoffs scores and each town's cutoffs vary from year to year. Using a wealth of variation of nearly 2,000 cutoffs from the high school assignment in Romania, PEU provides rigorous evidence of the impacts of going to a better school on the academic performance of students and on the behavior of parents and teachers. The economic logic of their application can be briefly summarized as follows.² Suppose there is a central planner who assigns students to high schools based on the students' scores on a placement test. High schools have limited capacity and can be ranked by some measure of quality. The central planner ranks students by their scores and assigns each of them to the best school available. Each student i submits her score X_i (forcing variable) to the central planner who based on the entire distribution of scores determines a minimum test score c_j (cutoff or

¹In a RDD setting with multiple cutoffs and treatments, it is unreasonable to expect that different local treatment effects are always identical. For example, Pop-Eleches and Urquiola (2013) finds that the impact of going to a better high school on academic achievement is heterogeneous across students with different ability levels. Another example is De La Mata (2012) who finds that the eligibility for Medicaid benefits, which depends on your income being below a threshold, decreases the probability of having private health insurance more strongly for lower income thresholds.

²The purpose of the simple example in the paragraph above is to introduce the reader to the framework of RDD with multiple cutoffs. The details of PEU's application are described in our application section.

threshold) for admission to each high school j in each town.³ The quality of high school j is denoted d_j , and different school qualities expose students to different treatment doses across cutoffs c_j . We are interested in the treatment effect of high school quality on the student i 's academic achievement Y_i (outcome variable). This effect is not immediately identified because we never observe the same students attending high schools of different qualities. As the test score crosses an admission threshold c_j , the quality of the school the student attends changes from d_{j-1} to d_j . By comparing students with test scores just below the cutoff to students with scores just above the cutoff, RDD allows identification of the impact of school quality on the average academic achievement of those students with test scores equal to c_j . We denote such local treatment effect by the average $\beta_j = \mathbb{E}[Y_i(d_j) - Y_i(d_{j-1})|X_i = c_j]$, where $Y_i(d_j)$ is the potential academic achievement student i has if attending a high school of quality d_j , and $\mathbb{E}[\cdot]$ denotes expected value over the distribution of students. The two sources of heterogeneity for local treatment effects are the different cutoff values and changes in treatment doses across the different cutoffs.

PEU is a particularly illustrative application because it exhibits sufficient variation in cutoff and treatment doses to generate ATEs with substantially greater economic relevance than the typical average based on normalizing all of the cutoffs to zero. Nevertheless, there are numerous other examples of RDD with many thresholds and heterogeneous treatments. For instance, Saavedra (2008) studies admission into more or less elite colleges where each college has its own cutoff that varies by year. Angrist and Lavy (1999) and Hoxby (2000) use class size rules to estimate the impact of class size on student achievement. In Hoxby (2000), the variation in cutoff values arises from specific school district class size rules. Several studies exploit different school starting dates to estimate the impact of educational attainment on various outcomes, e.g. Dobkin and Ferreira (2010), McCrary and Royer (2011). Duflo, Dupas, and Kremer (2011) analyzes school cohorts that are split into low and high-achieving

³For the sake of exposition, we assume for now that students attend the best high school available to them based on their score and the cutoffs that apply to them. In fact, some students may choose to attend a lesser high school so that we have a fuzzy rather than a sharp RDD. We examine the fuzzy case later in the paper.

classes based on test scores, where each school has its own cutoff score. Garibaldi, Giavazzi, Ichino, and Rettore (2012) looks at different income cutoffs that determine tuition subsidies to study the impact of tuition payment on the probability of late graduation from university.

The rapid growth in the number of applications of RDD in Economics in the late 1990s was accompanied by substantial theoretical contributions for inference in the one-cutoff case. Identification both in the sharp and fuzzy cases were formalized by Hahn, Todd, and Van der Klaauw (2001) who proposed estimation by local linear regression and derived its asymptotic distribution. Local polynomial regressions are known for low order bias at boundary points and rate optimality (Fan and Gijbels (1996), Porter (2003)). Recent theoretical contributions have addressed the optimal bandwidth choice (Imbens and Kalyanaraman (2012)); alternative asymptotic approximations with better finite sample properties (Cattaneo, Calonico, and Titiunik (2014)); quantile treatment effects (Frandsen, Frölich, and Melly (2012)) and kink treatment effects (Dong (2014)).

More closely related to our contribution is the study of treatment effect extrapolation, e.g. Angrist (2004), Bertanha and Imbens (2014), Angrist and Rokkanen (2013), and Rokkanen (2014). These last two papers use observations on additional covariates and restrict the relation between the heterogeneity of treatment effects and these covariates to obtain identification away from the cutoff. Our results differ from theirs because we use the variation of multiple cutoffs and doses to identify ATEs over distributions of individuals away from cutoffs without restricting the heterogeneity of treatment effects. In short, there are many applications with variation in cutoffs and treatment doses, but a lack of theoretical investigation on how to combine observations from all cutoffs to estimate economically relevant average effects.

The ability to combine different local effects to estimate an average effect depends on how comparable the researcher believes these effects are. We consider three cases of heterogeneity assumptions researchers could make about the comparability of treatment effects. The cases are presented in increasing order of imposed structure. In the first case of hetero-

ogeneity assumptions, the researcher does not believe changes in treatment doses $d_{j-1} \rightarrow d_j$ are quantitatively comparable across cutoffs c_j . This would be the case in the high school assignment example, if the quality of different schools cannot be credibly summarized in one metric d . Another example is the RDD setup studied in Hastings, Neilson, and Zimmerman (2013), where different score cutoffs assign students to different degree programs in Universities in Chile. One cutoff could switch students from Physics to Engineering and a second cutoff from Engineering to Economics, and it is difficult to summarize these treatment dose changes in one metric.

In the second case of heterogeneity assumptions, the researcher believes that treatment doses are quantitatively comparable across cutoffs, and treatment effects vary smoothly with changes in treatment dose. In the high school assignment example, PEU measures high school quality using average student performance in each school. They find behavioral evidence that parents and teachers perceive the schools' quality based on this measure. Another example of multiple cutoffs with comparable treatment doses is the case where there is one type of treatment triggered by one cutoff, but this one cutoff varies across subpopulations (e.g. towns, years, states). This is the case of Medicaid benefits for children studied in De La Mata (2012), where each state (subpopulation) has its own income threshold for Medicaid coverage eligibility (one treatment).

In the third case of heterogeneity assumptions, the researcher is willing to specify a parametric functional form for the treatment effect function $\beta(x, d, d') = \mathbb{E}[Y_i(d') - Y_i(d) | X_i = x]$. Economic theory or a priori knowledge guides the choice of a functional form that credibly summarizes the heterogeneity of treatment effects. In the high school assignment example, researchers could assume that returns to better education follow a polynomial function of scores and school quality to test for varying marginal returns. In a class size application like Hoxby (2000), a researcher could impose a functional form based on Lazear (2001)'s model of achievement as a function of class size.

This paper proposes consistent and asymptotically normal estimation procedures for

weighted average treatment effects. The interpretation of the average effect and the estimation procedure depends on the case of heterogeneity assumptions. In the first case of heterogeneity assumptions, the average effect is a weighted average of local treatment effects (ALTE) computed at the finite number of cutoffs observed in the data. Such an ALTE is a meaningful summary measure for counterfactual scenarios that weight individuals local to the existing cutoffs. In the high school assignment example, suppose one town decides to marginally expand capacity of some of its best high schools. Students that are currently near the admission cutoffs of such schools will be granted access to better school quality. The relative proportion of capacity increase across targeted high schools produces the relevant weighting scheme for the ALTE of this policy.

In the second case of heterogeneity assumptions, the treatment effects are described by a smooth function of cutoffs and doses. The average treatment effect (ATE) is a weighted integral of the treatment effect function $\beta(\cdot)$ over a set that envelops the variation of cutoffs and dose changes observed in the data. Such an ATE is a meaningful summary measure for counterfactual scenarios that weight individuals not only at the existing cutoffs but also between cutoffs. For example, suppose a new ‘charter’ school in a given town admits students by randomly drawing from a target population. This target population has a distribution of scores within the support of the observed distribution of scores in the data. The relevant ATE averages over the entire distribution of students that are granted access to this higher quality charter school, which includes scores between the existing cutoff values. Finally, in the third case of heterogeneity assumptions, interest lies not only in the ATE, which is the weighted integral of the parametric treatment effect function, but also on the parameters of this function. For example, a polynomial functional form on scores and average peer performance can be used to test the hypothesis that the returns of going to a better school is constant over scores.

The estimation procedure for the ALTEs and ATEs consists of two steps. The first step is identical in all three cases of heterogeneity assumptions. In the first step, we estimate

the local treatment effects at each cutoff non-parametrically by local polynomial regression. The second step depends on the case of heterogeneity assumptions the researcher is willing to make. In the first case of heterogeneity assumptions, the second step estimate for the ALTE is a simple weighted average of first step estimates, where the researcher chooses the weighting scheme. In the second case of heterogeneity assumptions, the second step estimation consists of estimating the function $\beta(x, d, d')$ non-parametrically using a local polynomial regression of the first step estimates on cutoff values (x, d, d') . The estimator for the ATE is simply the weighted integral of the estimated function $\hat{\beta}(x, d, d')$ over a set of values (x, d, d') that envelops the finite set of observed cutoff values. The researcher specifies the weighting density over this set according to the relevant counterfactual scenario.

A key contribution of this paper is the inference on the ATE in RDD when the treatment effect function is unknown and of infinite dimension. Consistency for the integral ATE requires an asymptotic sequence where both the number of observations and cutoffs go to infinity. Our estimator for the ATE is asymptotically normal and its maximum convergence rate is root-n. Lastly, in the third case of heterogeneity assumptions, estimates for the parametric functional form are obtained by regressing the first step estimates on functions of cutoff-dose values specified by the researcher. Observations from different cutoffs are optimally weighted in this second step regression depending on the first step variance. The ATE in the third case of heterogeneity assumptions is the weighted integral of the estimated parametric functional form. We show consistency and asymptotic normality of all estimators.

Another advantage of the third case of heterogeneity assumptions is that a parametric functional form gives identification in the fuzzy RDD case. In the high school assignment example, when a student is accepted for a high school with peer-quality d , she may choose to attend a different high school d' in the fuzzy case. When there are many cutoffs, each cutoff may exhibit its own compliance behavior. The observed average outcome of students around cutoff c is a weighted average of potential outcomes $Y_i(d)$ for the different school qualities d students choose to attend when their test score is near c . Thus, comparing the academic

performance of students just below the cutoff to those just above gives a mix of treatment effects of various doses.

To disentangle the different treatment effects, we make a minimal assumption about how the behavior of students changes with their test score by ruling out ‘defiance’: if the test score of a student currently attending high school B increases so as to grant her access to high school A, we assume she either chooses to attend school A or stay at school B, and that she is not triggered to attend some other school C. When she chooses school A, we say she complies to the treatment eligibility change associated with the cutoff of school A. We call a student ‘complier’ when she complies to the treatment change for at least one of the cutoffs and does not respond to the treatment change of other cutoffs. No-defiance is not a sufficient condition for identification of treatment effects on compliers in all cutoffs. For example, suppose we have a town with three schools and two cutoffs. At the second cutoff that grants admission to the best school, there could be compliers that change from the worst school into the best school, and from the second best to the best school. We only observe the average change in the outcome variable aggregated over these two types of compliers and cannot separately identify their treatment effects without further functional form assumptions. The parametric functional form and no-defiance are sufficient conditions for identification of treatment effects on compliers. We provide a two-step estimator for such a parametric functional form and show consistency and asymptotic normality.

The remainder of this paper is divided into three main sections for each case of heterogeneity assumptions. Section 2 sets up the notation for RDD with multiple cutoffs to be used in all sections, describing the estimation procedure and providing sufficient conditions for inference on the ALTE in the first case of heterogeneity assumptions. Section 3 derives an estimator for the ATE and lays down sufficient conditions for valid inference in the second case of heterogeneity assumptions. Section 4 explains the estimation procedure for the parameters of a functional form specified by the researcher in the third case of heterogeneity assumptions; sufficient conditions are given for the asymptotic inference in both the sharp

and fuzzy RDD cases. Section 5 illustrates our methods using data from PEU. Section 6 concludes. The proofs are found in the Appendix.⁴

2 Average of Local Treatment Effects

In this section we assume the sharp RDD case and that the researcher believes that changes in treatment doses do not bear any quantitative relationship across cutoffs. Individuals are subject to different treatments across cutoffs, but the treatment dose cannot be summarized into a scalar variable. First, we define the notation for the sharp RDD with multiple thresholds. Second, we re-state an already known identification result in terms of framework of multiple thresholds. Third, we define an average treatment effect (ATE) parameter where the researcher chooses the weighting scheme. Fourth, we describe an estimation procedure for this ATE and show consistency and asymptotic normality when the sample size grows large but the number of cutoffs is held fixed.

There are many cutoffs c defined on one scalar forcing variable X that assign individuals to different treatment doses defined by the variable D . In this section, the variable D can be some qualitative measure of the treatment received. This assignment mechanism is observed for different sub-populations of individuals where the cutoffs and doses vary according to the sub-population. In the example of high school assignment, a sub-population is a town-year. Random variables for each individual are indexed by ‘i’. The forcing variable of individual ‘i’ is denoted by X_i and lives in a compact interval $\mathcal{X} = [\underline{\mathcal{X}}, \overline{\mathcal{X}}]$. The set of possible treatment doses is defined as $\mathcal{D} = [\underline{\mathcal{D}}, \overline{\mathcal{D}}]$, also a compact interval. The discrete variable P_i takes values in $\mathcal{P} = \{1, \dots, P\}$ and indicates the sub-population of individual ‘i’. In sub-population $p \in \mathcal{P}$, each cutoff $c_{p,j}$ is indexed by $j \in \mathcal{J}_p = \{1, \dots, K(p)\}$ where $K(p)$ is total number of cutoffs in sub-population p and $c_{1,p} < c_{2,p} < \dots < c_{K(p),p}$. In this section the assignment is sharp which means that an individual with forcing variable X_i in sub-population P_i is deterministically assigned to treatment dose $D_i = D(X_i, P_i)$ for some known assignment

⁴The Appendix is available online at www.stanford.edu/~bertanha/Bertanha_JMP_appendix.pdf

mapping $D : \mathcal{X} \times \mathcal{P} \rightarrow \mathcal{D}$ where

$$D(x, p) = \begin{cases} d_{p,0} & \text{if } c_{p,0} \leq x < c_{p,1} \\ d_{p,1} & \text{if } c_{p,1} \leq x < c_{p,2} \\ \vdots & \\ d_{p,K(p)} & \text{if } c_{p,K(p)} \leq x \leq c_{p,K(p)+1} \end{cases} \quad (1)$$

with $c_{0,p} = \underline{\mathcal{X}}$, $c_{K(p)+1,p} = \overline{\mathcal{X}}$. The schedule of cutoffs and treatment doses in each sub-population $p \in \mathcal{P}$ is given by the non-random set $\{c_{p,j}, d_{p,j-1}, d_{p,j}\}_{j \in \mathcal{J}_p}$.⁵ The total number of cutoffs from all sub-populations is $K = \sum_{p \in \mathcal{P}} K(p)$.⁶

We follow the modern literature on treatment effects and use Rubin's model of potential outcomes (Rubin (1974), Imbens and Lemieux (2008)). The potential outcome for individual 'i' if she receives treatment dose 'd' is denoted as the random variable $Y_i(d)$. The data generating process can be summarized as follows. Values for the forcing variable X_i , sub-population P_i and potential outcomes $\{Y_i(d)\}_{d \in \mathcal{D}}$ are drawn iid $i = 1, \dots, n$ from some joint distribution. Given the mapping $D(x, p)$, these n individuals are assigned to different treatment doses $D_i = D(X_i, P_i)$. The observed outcome Y_i is given by

$$Y_i = \sum_{p \in \mathcal{P}} \sum_{j \in \mathcal{J}_p^0} Y_i(d_{p,j}) \mathbb{I}\{D_i = d_{p,j}, P_i = p\}$$

where $\mathcal{J}_p^0 = \mathcal{J}_p \cup \{0\} = \{0, 1, \dots, K(p)\}$, and $\mathbb{I}\{\cdot\}$ is the indicator function.

⁵The validity of the RDD depends crucially on the exogeneity of cutoffs and no manipulation of the forcing variable X by individuals. See McCrary (2008) for a test of forcing variable manipulation. Bajari, Hong, Park, and Town (2011) presents a modified RDD estimator that is consistent under forcing variable manipulation in a class of structural models.

⁶Each cutoff $c_{p,j}$ has a change in treatment dose $d_{p,j} \neq d_{p,j-1}$. Note that the assignment function and schedule of cutoffs defined above accommodates different deterministic assignment schemes based on cutoffs for one forcing variable. In the high school assignment example, the ranking of admission cutoffs may not correspond to the ordering of school qualities if students are allowed to attend a school different than the initial assignment. Our definition of cutoffs above can fit both the case where a student attends the best high school into which she is admitted or the high school with the highest admission cutoff that is lower than her test score. The number of cutoffs that corresponds to a change in treatment dose is different in each case.

The econometrician observes the schedule of cutoffs and treatment doses for all sub-populations and (Y_i, X_i, D_i, P_i) for $i = 1, \dots, n$. For individuals with a given value for the forcing-variable and sub-population, the treatment effect is the average change in outcome due to a change in treatment dose for these individuals. RDD identifies these treatment effects for each cutoff value c of the forcing-variable in each sub-population p and for the change in treatment dose $d \rightarrow d'$ associated with that cutoff. This treatment effect is denoted as

$$\beta(c, d, d', p) = E[Y_i(d') - Y_i(d) | X_i = c, P_i = p] \quad (2)$$

It is widely known in the RDD literature that continuity of the conditional mean of potential outcomes lead to identification of treatment effects at the cutoff values (Hahn, Todd, and Van der Klaauw (2001)). This is re-stated in lemma 1 for the multiple cutoff case.

Lemma 1. *For any $p \in \mathcal{P}$, $j \in \mathcal{J}_p$, and $d \in \mathcal{D}$, assume $E[Y_i(d) | X_i = x, P_i = p]$ is a continuous function of x . Then, the treatment effects $\beta(c_{p,j}, d_{p,j-1}, d_{p,j}, p)$ for every $p \in \mathcal{P}$ and $j \in \mathcal{J}_p$ are identified:*

$$\beta(c_{p,j}, d_{p,j-1}, d_{p,j}, p) = \lim_{e \downarrow 0} \{E[Y_i | X_i = c_{p,j} + e, P_i = p] - E[Y_i | X_i = c_{p,j} - e, P_i = p]\}$$

The goal of this paper is to exploit the variation in treatment effects that arises from different cutoff-dose values. To combine treatments effects from various sub-populations we make the following assumption

Assumption 1. *For any $d', d \in \mathcal{D}$, $x \in \mathcal{X}$, and $p \in \mathcal{P}$*

$$E[Y_i(d') | X_i = x, P_i = p] - E[Y_i(d) | X_i = x, P_i = p] = \beta(x, d, d')$$

This assumption says that individuals with the same observed forcing variable X_i that undergo the same change in treatment dose $d \rightarrow d'$ have the same average response across different sub-populations. It does not restrict conditional means to be same across different sub-populations which accommodates time-trends and sub-population fixed effects for example.

There are two sources of heterogeneity for treatment effects across different cutoffs. Treatment effects are expected to be heterogeneous because of different values of the forcing-variable and changes in treatment doses. Researchers are often interested in summary measures of the different treatment effects. In this paper we work with average treatment effect as a default summary measure. A common practice in applied work is to normalize all cutoffs to zero in order to use existing estimation techniques for the one-cutoff case. This procedure produces an estimate for an average effect weighted by unknown coefficients. An average treatment effect is only informative when the researcher deliberately chooses how different treatment effects are weighted. For example, a certain policy may have positive effects for some values of the forcing variable but negative effects for other values. Depending on how these effects are weighted, we may conclude the policy has no effect in a given population.

Existing data can be used to estimate an average treatment effect of the current policy but also new policies. Each counterfactual scenario translates into a weighting scheme over the observed cutoffs $(c_{p,j}, d_{p,j-1}, d_{p,j})$. For example, a new policy may re-allocate students across the existing schools. The choice of weighting scheme is such that $\omega_{p,j}$ represents the probability mass of students with test score close to $c_{p,j}$ that undergo a change in school quality of $d_{p,j-1} \rightarrow d_{p,j}$. We define the average of local treatment effects (ALTE) for a set of weights $\{\omega_{p,j}\}_{p \in \mathcal{P}, j \in \mathcal{J}_p}$ as

$$\mu_{ALTE} = \sum_{p \in \mathcal{P}, j \in \mathcal{J}_p} \omega_{p,j} \beta(c_{p,j}, d_{p,j-1}, d_{p,j})$$

The average treatment effect μ_{ALTE} is identified under continuity of the conditional mean of potential outcomes (Lemma 1) for any choice of weighting scheme. Note that the set of counterfactuals for which this ALTE has meaning is restricted to the finite set of observed cutoff-dose values. In the second case of heterogeneity assumptions, it is possible to define an ATE for a much larger set of counterfactuals.

In what follows, we describe a two-step estimation procedure for μ_{ALTE} . The main idea of the first step is to use observations near each of the cutoffs $c_{p,j}$ to estimate the change in conditional mean of Y_i given X_i, P_i at cutoff $c_{p,j}$. The main idea of the second step is average out these estimates across the different cutoffs $c_{p,j}$ using the weights $\omega_{p,j}$. In the first step, we obtain a non-parametric estimate of $B_{p,j}$ where

$$B_{p,j} = \lim_{e \downarrow 0} \{E[Y_i|X_i = c_{p,j} + e, P_i = p] - E[Y_i|X_i = c_{p,j} - e, P_i = p]\} \quad (3)$$

at each cutoff $c_{p,j}$ using local polynomial regression (LPR). By lemma 1, $B_{p,j} = \beta(c_{p,j}, d_{p,j-1}, d_{p,j}) \equiv \beta_{p,j}$. The researcher chooses a bandwidth parameter $h > 0$, a kernel density function $k(\cdot)$, and the order of the polynomial regression $\rho \in \mathbb{Z}_+$.⁷ The bandwidth parameter defines a neighborhood around each cutoff from which we use observations in the estimation. The farther observations are from the cutoff, the less weight they receive which is determined by the function $k(\cdot)$. We fit a polynomial in X on each side of the cutoff, and the estimator $\hat{B}_{p,j}$ is the difference between the intercept of these two polynomial regressions.

⁷Common choices in the applied literature for these are the edge kernel $k(u) = \mathbb{I}\{|u| \leq 1\}(1 - |u|)$, $\rho = 1$ (local linear regression), and the optimal bandwidth proposed by Imbens and Kalyanaraman (2012).

$$\hat{B}_{p,j} = \hat{a}_{p,j}^+ - \hat{a}_{p,j}^- \quad (4)$$

$$(\hat{a}_{p,j}^+, \hat{b}_{p,j}^+) = \underset{(a,b)}{\operatorname{argmin}} \sum_{i=1}^n k\left(\frac{X_i - c_{p,j}}{h}\right) v_i^{p,j+} [Y_i - a - b_1(X_i - c_{p,j}) - \dots - b_\rho(X_i - c_{p,j})^\rho]^2 \quad (5)$$

$$(\hat{a}_{p,j}^-, \hat{b}_{p,j}^-) = \underset{(a,b)}{\operatorname{argmin}} \sum_{i=1}^n k\left(\frac{X_i - c_{p,j}}{h}\right) v_i^{p,j-} [Y_i - a - b_1(X_i - c_{p,j}) - \dots - b_\rho(X_i - c_{p,j})^\rho]^2 \quad (6)$$

and $v_i^{p,j+} = \mathbb{I}\{c_{p,j} \leq X_i < c_{p,j} + h, P_i = p\}$, $v_i^{p,j-} = \mathbb{I}\{c_{p,j} - h < X_i < c_{p,j}, P_i = p\}$.

The LPR estimator is known in the literature for its nice boundary properties and rate optimality (Fan and Gijbels (1996), Porter (2003)). In the second step, the researcher averages out $\hat{B}_{p,j}$ to obtain the estimator $\hat{\mu}_{ALTE}$ for μ_{ALTE} .

$$\hat{\mu}_{ALTE} = \sum_{p \in \mathcal{P}, j \in \mathcal{J}_p} \omega_{p,j} \hat{B}_{p,j} \quad (7)$$

For the case of one sub-population, one cutoff, the asymptotic distribution of the RDD treatment effect has been derived by Porter (2003). Minor adjustments give the distribution of each $\hat{B}_{p,j}$ and the weighted average. Asymptotic normality requires $n \rightarrow \infty$ and $h \rightarrow 0$. Since the number of cutoffs K is fixed and $h \rightarrow 0$, the neighborhoods around each cutoff don't overlap for large n . In large samples, each individual observation is used for only one $\hat{B}_{p,j}$ which makes $\hat{B}_{p,j} \perp \hat{B}_{p,l}$ for $j \neq l$. Therefore, the asymptotic distribution of $\hat{\mu}_{ALTE}$ will be the weighted sum of the asymptotic distributions of each $\hat{B}_{p,j}$. Below, we list sufficient conditions for the asymptotic normality result in theorem 1.

Assumption 2. *The kernel density function $k : \mathbb{R} \rightarrow \mathbb{R}$ is symmetric, bounded, has compact support $[-M, M]$, and can be written as the difference of two weakly increasing functions on \mathbb{R} .*

Assumption 3. (i) For every $p \in \mathcal{P}$, $X_i|P_i = p$ has probability density function $f_{X|P}(x, p)$ and a bounded support $\mathcal{X} = [\underline{\mathcal{X}}, \overline{\mathcal{X}}]$; $f_{X|P}(x, p)$ is bounded and bounded away from zero uniformly on $(x, p) \in \mathcal{X} \times \mathcal{P}$.

(ii) $f_{X|P}(x, p)$ is one time differentiable w.r.t x with partial derivative $\nabla_x f_{X|P}(x, p)$ bounded on $(x, p) \in \mathcal{X} \times \mathcal{P}$.

(iii) $\forall p \in \mathcal{P}, \mathbb{P}(P_i = p) = q_p > 0$

Assumption 4. Let $\rho \in \mathbb{Z}_+$ be the order of the LPR.

(a) $R(x, d, p) = E[Y_i(d)|X_i = x, P_i = p]$ is $\rho + 1$ times continuously differentiable w.r.t. x with $\rho + 1$ -th partial derivative $\nabla_x^{\rho+1} R(x, d, p)$

(b) $\sigma^2(x, d, p) = E \{ [Y_i(d) - R(x, d, p)]^2 | X_i = x, P_i = p \}$ is one time continuously differentiable w.r.t. x with partial derivative $\nabla_x \sigma^2(x, d, p)$

Theorem 1. Suppose assumptions 1, 2, 3, 4 hold. As $n \rightarrow \infty$ and $h \rightarrow 0$, assume $nh \rightarrow \infty$ and $\sqrt{nh}h^{\rho+1} \rightarrow C \in [0, \infty)$. Then,

$$\sqrt{nh} (\hat{\mu}_{ALTE} - \mu_{ALTE}) \xrightarrow{d} N \left(C \sum_{p,j} \omega_{p,j} \mathcal{B}_{p,j}; \sum_{p,j} \omega_{p,j}^2 \mathcal{V}_{p,j} \right)$$

where

$$\mathcal{B}_{p,j} = \frac{1}{(\rho+1)!} [\nabla_x^{\rho+1} m(c_{p,j}^+, p) - (-1)^{\rho+1} \nabla_x^{\rho+1} m(c_{p,j}^-, p)] e_1' \Gamma^{-1} \gamma^* \quad (8)$$

$$\mathcal{V}_{p,j} = \frac{\zeta^2(c_{p,j}^+, p) + \zeta^2(c_{p,j}^-, p)}{f_{X|P}(c_{p,j}, p) q_p} e_1' \Gamma^{-1} \Delta \Gamma^{-1} e_1 \quad (9)$$

$$\nabla_x^{\rho+1} m(c_{p,j}^+, p) = \lim_{x \downarrow c_{p,j}} \nabla_x^{\rho+1} E[Y_i | X_i = x, P_i = p]$$

$$\nabla_x^{\rho+1} m(c_{p,j}^-, p) = \lim_{x \uparrow c_{p,j}} \nabla_x^{\rho+1} E[Y_i | X_i = x, P_i = p]$$

$$\zeta^2(c_{p,j}^+, p) = \lim_{x \downarrow c_{p,j}} E \{ (Y_i - E[Y_i | X_i, P_i])^2 \mid X_i = x, P_i = p \}$$

$$\zeta^2(c_{p,j}^-, p) = \lim_{x \uparrow c_{p,j}} E \{ (Y_i - E[Y_i | X_i, P_i])^2 \mid X_i = x, P_i = p \}$$

$$\Gamma = \begin{bmatrix} \gamma_0 & \cdots & \gamma_\rho \\ \vdots & \vdots & \vdots \\ \gamma_\rho & \cdots & \gamma_{2\rho} \end{bmatrix} \text{ and } \Delta = \begin{bmatrix} \delta_0 & \cdots & \delta_\rho \\ \vdots & \vdots & \vdots \\ \delta_\rho & \cdots & \delta_{2\rho} \end{bmatrix}$$

$$\gamma^* = [\gamma_{\rho+1} \quad \cdots \quad \gamma_{2\rho+1}]'$$

$$e_1 \text{ is the } (\rho + 1 \times 1) \text{ vector } e_1 = [1 \ 0 \ 0 \ \cdots \ 0]'$$

$$\gamma_d = \int_0^1 k(u) u^d du \text{ and } \delta_d = \int_0^1 k(u)^2 u^d du$$

ρ is the order of the LPR

To perform inference using this asymptotic result, we need consistent estimators for the asymptotic bias and variance in equations 8 and 9. The researcher chooses ρ and the kernel density function $k(\cdot)$ which give γ^* , Γ and Δ ; the bandwidth value can be used to infer $\hat{C} = \sqrt{n h h^{\rho+1}}$. It remains to estimate $\nabla_x^{\rho+1} m(c_{p,j}^\pm, p)$, $\zeta^2(c_{p,j}^\pm, p)$, and $f_{X|P}(c_{p,j}, p) q_p$ which is a straightforward non-parametric problem. For the side derivatives $\nabla_x^{\rho+1} m(c_{p,j}^\pm, p)$, a consistent estimator is obtained from a LPR of order $\rho + 1$ (equations 5 and 6) that uses observations from each side of the cutoff $c_{p,j}$. The estimator is simply the slope coefficient on $(x - c_{p,j})^{\rho+1}$. Lemma 7 in the Appendix shows consistency of this estimator. The density

$f_{X|P}(c_{p,j}, p)q_p$ is consistently estimated (Silverman (1986)) by

$$\hat{f}_{X,P}(c_{p,j}, p) = \frac{1}{nh} \sum_{i=1}^n k \left(\frac{X_i - c_{p,j}}{h} \right) \mathbb{I}\{P_i = p\}$$

Porter (2003) suggests the following consistent estimation procedure for the side limits of the variance $\zeta^2(c_{p,j}^\pm, p)$:

$$\begin{aligned} \hat{m}(x, p) &= \frac{\frac{1}{nh} \sum_{i=1}^n k \left(\frac{X_i - c_{p,j}}{h} \right) \mathbb{I}\{P_i = p\} \left(Y_i - \sum_{j \in \mathcal{J}_p} \mathbb{I}\{c_{p,j} \leq x\} \hat{B}_{p,j} \right)}{\hat{f}_{X,P}(c_{p,j}, p)} \\ \hat{\varepsilon}_i &= Y_i - \hat{m}(X_i, P_i) - \sum_{j \in \mathcal{J}_p} \mathbb{I}\{c_{p,j} \leq x\} \hat{B}_{p,j} \\ \hat{\zeta}^2(c_{p,j}^+, p) &= \frac{\frac{1}{nh} \sum_{i=1}^n v_i^{p,j+} k \left(\frac{X_i - c_{p,j}}{h} \right) \hat{\varepsilon}_i^2}{\frac{1}{2} \hat{f}_{X,P}(c_{p,j}, p)} \\ \hat{\zeta}^2(c_{p,j}^-, p) &= \frac{\frac{1}{nh} \sum_{i=1}^n v_i^{p,j-} k \left(\frac{X_i - c_{p,j}}{h} \right) \hat{\varepsilon}_i^2}{\frac{1}{2} \hat{f}_{X,P}(c_{p,j}, p)} \end{aligned}$$

Using these estimators along with equations 8 and 9 and definitions of theorem 1, we obtain estimators for the asymptotic variance and bias of $\hat{\mu}_{ALTE}$. Alternative approaches to compute the standard errors include bootstrapping (Hardle and Bowman (1988), Neumann and Polzehl (1998)), and using empirical likelihood methods of Otsu, Xu, and Matsushita (2014) to produce valid confidence intervals without having to estimate these asymptotic objects.

3 General Average Treatment Effects

In the second case of heterogeneity assumptions, we assume the sharp RDD case and that the researcher finds reasonable to summarize high school j 's quality using a scalar variable d_j . Moreover, the treatment effect β_j is described by a smooth function $\beta(\cdot)$ of the admission cutoffs c_j and treatment doses (d_{j-1}, d_j) associated with cutoff j . Examples of measures

of school quality include the average test score of peers, the average number of teachers or funding per student. Using the treatment dose d_j , different treatments can be related to each other quantitatively. Provided we have sufficient variation in cutoff-doses across sub-populations, these heterogeneity assumptions allows inference on an ATE computed over a set of cutoff-doses values that envelops the finite set of cutoff-doses observed. We start this section by defining such ATE parameter where the researcher chooses a weighting scheme. Then, we describe an estimation procedure for this ATE and show consistency and asymptotic normality when the sample size and the number of cutoffs grows large.

In the first case of heterogeneity assumptions, the set of counterfactuals for which we can compute the ATE is limited to the finite set of cutoff-doses observed. It is unlikely that new policies will have the same schedule of cutoffs and treatment doses as the policy we observe in the data. The ability to compute weighted effects for cutoff and dose values beyond the existing ones thus becomes crucial. Provided we have sufficient variation of cutoff and dose changes, we can also define an average treatment effect that goes beyond the finite number of cutoff-doses. For example, we may be interested in the average impact of increasing everyone's treatment dose by 10%. The average effects $\beta(x, d, d')$ should be weighted by the joint density of $(X, D, D') = (X, D, 1.1D)$.

We define the set \mathcal{C} to be a compact subset of $\mathcal{X} \times \mathcal{D} \times \mathcal{D}$ that contains the finite set of cutoffs and dose changes we observe in the data. The set \mathcal{C} is specified by the researcher to be set that envelops the variation of cutoffs and dose values of in the data. In other words, if there were an infinite amount of data (cutoffs), the chosen set \mathcal{C} is believed to be dense in the infinite set of cutoffs and dose values. For example, we can choose \mathcal{C} to be the convex-hull of the finite set $\{c_{p,j}, d_{p,j-1}, d_{p,j}\}$.

The researcher specifies an integrable weighting function $\omega : \mathcal{C} \rightarrow \mathbb{R}$ which defines an ATE parameter μ_{ATE} computed over the set \mathcal{C} .

$$\mu_{ATE} = \int_{\mathcal{C}} \omega(x, d, d') \beta(x, d, d') d(x, d, d')$$

The ATE parameter μ_{ATE} is identified as long as we have an infinite amount of data for infinitely many cutoff-dose values that cover the set \mathcal{C} .

Lemma 2. *Assume*

- $\omega(x, d, d')$, $\beta(x, d, d')$ are continuous in the compact set $\mathcal{C} \subset \mathbb{R}^3$
- the population version of the model (infinite amount of data) is as following: there is an infinite countable set of cutoff-doses \mathcal{C}^* that is dense in \mathcal{C} such that $\beta(c, d, d')$ is identified for every $(c, d, d') \in \mathcal{C}^*$

Then, μ_{ATE} is identified.

The first step of the estimation procedure for μ_{ATE} is identical to the first step of the estimation of μ_{ALTE} . The difference is the estimation of the infinite dimensional object $\beta(x, d, d')$ in the second step. In the first step, LPRs produce estimates for each $B_{p,j}$ using observations in the neighborhood of each cutoff p, j in the data (equations 4, 5 and 6). In the second step, we use multivariate local polynomial regression of $\widehat{B}_{p,j}$ on $\{c_{p,j}, d_{p,j-1}, d_{p,j}\}$ to construct an estimate $\widehat{\beta}(x, d, d')$ for the function $\beta(x, d, d')$. The estimator $\widehat{\mu}_{ATE}$ is the weighted integral of $\widehat{\beta}(x, d, d')$. The researcher chooses a bandwidth $h_I > 0$, the maximum degree of the polynomials $\rho_I \geq 3$, and a kernel density function $k(u)$. For each point $(x, d, d') \in \mathcal{C}$, the estimate $\widehat{\beta}(x, d, d')$ is obtained by the following least squares minimization:

$$\widehat{\beta}(x, d, d') = \widehat{\eta}_1 \tag{10}$$

$$\widehat{\eta} = \underset{\eta}{\operatorname{argmin}} \left(\widehat{B} - E(x, d, d')\eta \right)' \Omega_{h_I}(x, d, d') \left(\widehat{B} - E(x, d, d')\eta \right) \tag{11}$$

where $E(x, d, d')$ is the $K \times J$ matrix formed by stacking the $K \times 1$ vectors $E_{p,j}(x, d, d')$; η is a $J \times 1$ vector of parameters; each $E_{p,j}(x, d, d')$ is a vector of the standard basis of the space of polynomials in \mathbb{R}^3 of maximum degree ρ_I evaluated at $((x - c_{p,j}), (d - d_{p,j-1}), (d' - d_{p,j}))$; each entry in $E_{p,j}(x, d, d')$ is a polynomial of the form $p(x, d, d') = (x - c_{p,j})^{\gamma_1} (d - d_{p,j-1})^{\gamma_2} (d' - d_{p,j})^{\gamma_3}$ with $\gamma_1, \gamma_2, \gamma_3 \in \{0, 1, \dots, \rho_I\}$ $\gamma_1 + \gamma_2 + \gamma_3 \leq \rho_I$; the first entry in $E_{p,j}(x, d, d')$ is the polynomial of degree zero (i.e. $p(x, d, d') = 1$), the next 3 entries are all the polynomials of degree 1 (i.e. all $p(x, d, d')$ such that $\gamma_1 + \gamma_2 + \gamma_3 = 1$), and then all the polynomials of degree 2, and so on, until degree ρ_I ; therefore, $J = \binom{\rho_I+3}{3}$, and η_1 is the first coordinate of the vector η (intercept coefficient); $\Omega_{h_I}(x, d, d')$ is the $K \times K$ diagonal matrix of kernel weights:

$$\Omega_{h_I}(x, d, d') = \text{diag} \{ \Omega_{h_I,p,j} \}_{p,j} = \text{diag} \left\{ k \left(\frac{x - c_{p,j}}{h_I} \right) k \left(\frac{d - d_{p,j-1}}{h_I} \right) k \left(\frac{d' - d_{p,j}}{h_I} \right) \right\}_{p,j}$$

The estimator $\hat{\mu}_{ATE}$ is the weighted integral of $\hat{\beta}(x, d, d')$ over set \mathcal{C} with weighting density $\omega(x, d, d')$ chosen by the researcher. The integral $\hat{\mu}_{ATE}$ can be written as a finite weighted sum of the first stage estimates $\hat{B}_{p,j}$.

$$\hat{\mu}_{ATE} = \int_{\mathcal{C}} \omega(x, d, d') \hat{\beta}(x, d, d') d(x, d, d') = \sum_{p \in \mathcal{P}, j \in \mathcal{J}_p} \Delta_{p,j} \hat{B}_{p,j}$$

where $\Delta_{p,j}$ are called ‘corrected weights’. They are implicitly defined by the second step local polynomial regression and given by the formula

$$\begin{aligned}
\Delta_{p',j'} &= \int_{\mathcal{C}} \omega(x, d, d') e_1' (E(x, d, d')' \Omega_{h_I}(x, d, d') E(x, d, d'))^{-1} \\
&\quad \Omega_{h_I, p', j'}(x, d, d') E_{p', j'}(x, d, d') d(x, d, d') \\
&= \int_{\mathcal{C}} \omega(x, d, d') \frac{\det \left(E(x, d, d')' \Omega_{h_I}(x, d, d') E_{\mathbf{0} \leftarrow e_{p', j'}}(x, d, d') \right)}{\det (E(x, d, d')' \Omega_{h_I}(x, d, d') E(x, d, d'))} d(x, d, d') \quad (12)
\end{aligned}$$

where e_1 is the $J \times 1$ vector $e_1 = [1 \ 0 \ 0 \ \cdots \ 0]'$, and $E_{\mathbf{0} \leftarrow e_{p', j'}}(x, d, d')$ is the matrix valued function $E(x, d, d')$ except for the first column which is replaced by the $K \times 1$ vector $e_{p', j'}$ that is zero everywhere except for the (p', j') -th entry which is equal to 1.

A necessary condition for consistency of $\hat{\mu}_{ATE}$ is that the schedule of cutoff-dose values becomes dense in the set \mathcal{C} as $K \rightarrow \infty$. Our asymptotic exercise has the sample size $n \rightarrow \infty$, the total number of cutoffs $K \rightarrow \infty$, but the number of sub-populations P is fixed.⁸ The integral of a function can be approximated by the weighted sum of the values of such function at a finite number of points in its domain. The approximation error converges to zero as the number of points grows large. In our case, the function evaluations are estimated which means that the integral approximation error has to converge to zero fast enough to ensure asymptotic normality.

Assumption 5 states conditions on the limiting behavior of the schedule of cutoffs and treatment doses and on how it approximates set \mathcal{C} .

Assumption 5.

(a) *The schedule of cutoffs and doses comes from a triangular array of fixed constants that*

⁸Another less tractable asymptotic exercise could have $n \rightarrow \infty$, $P \rightarrow \infty$ but a fixed number of cutoffs $K(p)$ in each sub-population. This alternative asymptotics has the probability of some sub-populations converge to zero. Conditional mean estimators have these probabilities in the denominator which complicates their asymptotics, and we chose not to pursue this route. Moreover, our pooling assumption 1 makes an additional treatment effect in an existing sub-population indistinguishable than the same treatment effect in an additional sub-population.

depends on K (total number of cutoffs) denoted \mathcal{C}_K :

$$\mathcal{C}_K = \{(c_{p,j,K}, d_{p,j-1,K}, d_{p,j,K})\}_{p \in \mathcal{P}, j \in \mathcal{J}_{p,K}}$$

where $\mathcal{C}_K \subset \mathcal{C}$ and $\mathcal{J}_{p,K} = \{1, \dots, K_K(p)\}$ with total number of cutoffs in subpopulation p indexed by K ;

- (b) The maximum distance between two consecutive cutoffs within a subpopulation $p \in \mathcal{P}$ is inversely proportional to the total number of cutoffs $K(p)$ in that subpopulation

$$\max_{j=1, \dots, K_K(p)} |c_{p,j,K} - c_{p,j-1,K}| = O(K_K(p)^{-1})$$

- (c) given the second step estimation bandwidth sequence $h_{I,K}$ (which is such that $1/(Kh_{I,K}^3) = O(1)$) the number of points in \mathcal{C}_K that are within the $h_{I,K}$ neighborhood of any point in \mathcal{C} is of order $Kh_{I,K}^3$:

$$\sup_{(x,d,d') \in \mathcal{C}} \sum_{j=1}^K \mathbb{I} \{ \Omega_{h_{I,K},j}(x, d, d') > 0 \} = O(Kh_{I,K}^3)$$

where $\Omega_{h_{I,K},j}(x, d, d')$ is defined after equation 11;

- (d) for every $\rho_I \in \mathbb{Z}_+$ (degree of 2nd step local polynomial)

$$\sup_{\substack{(x,d,d') \in \mathcal{C} \\ 1 \leq j \leq K}} \left| \frac{\det(E(x, d, d')' \Omega_{h_{I,K}}(x, d, d') E_{\mathbf{0} \leftarrow e_j}(x, d, d'))}{\det(E(x, d, d')' \Omega_{h_{I,K}}(x, d, d') E(x, d, d'))} \right| = O((Kh_{I,K}^3)^{-1})$$

where $E(x, d, d')$ and $\Omega_{h_{I,K}}(x, d, d')$ were defined after equation 11; $E_{\mathbf{0} \leftarrow e_j}(x, d, d')$ is equal to $E(x, d, d')$ except for the first column which has the $K \times 1$ vector e_j which is zero everywhere except for j -th coordinate that is equal to 1.

Conditions in assumption 5 basically require two things: for large K , (i) the proportion of observations in the h_I neighborhood of any point in set \mathcal{C} is of the same order of the volume

of this neighborhood, that is, h_I^3 ; (ii) there is always enough points in the h_I neighborhood of any point in set \mathcal{C} for the invertibility of the $(E(x, d, d')'\Omega_{h_{I,K}}(x, d, d')E(x, d, d'))$ matrix. These conditions should be satisfied in a variety of examples of triangular arrays of points, and we give one simple example in the Appendix (section 7.9.3) where these conditions hold. Next, we state sufficient smoothness conditions on the functions $\omega(x, d, d')$ and $\beta(x, d, d')$.

Assumption 6. (a) $\omega(x, d, d')$ is a continuous function in (x, d, d') ;

(b) for $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{Z}_+^3$, let

$$\nabla^{|\gamma|}\beta(x, d, d') = \frac{\partial^{\gamma_1+\gamma_2+\gamma_3}}{\partial x^{\gamma_1}\partial d^{\gamma_2}\partial d'^{\gamma_3}}\beta(x, d, d')$$

denote the partial derivatives of $\beta(x, d, d')$. Assume $\nabla^{|\gamma|}\beta(x, d, d')$ is continuous for every γ such that $|\gamma| \leq \rho_I + 1$, where ρ_I is the polynomial degree in the second step estimation.

(c) $R(x, d, p) \equiv E[Y_i(d)|X_i = x, P_i = p]$ is $\rho + 2$ times continuous differentiable wrt x with $\rho + 2$ -th partial derivative $\nabla_x^{\rho+2}R(x, d, p)$ where ρ is the order of the LPR in the first step estimation

(d) $\nabla_x^{\rho+2}R(x, d, p)$ and $\nabla_x\sigma^2(x, d, p)$ are bounded functions of (x, d, p)

(e) $Y_i(d) - R(x, d, p)$ is a bounded random function of (x, d, p) a.s.

(f) For the schedule of cutoffs and doses \mathcal{C}_K and corrected weights $\Delta_{p,j,K}$ (defined in equation 12) the limits below are well defined.

$$\lim_{K \rightarrow \infty} \left\{ \sum_{p,j} \Delta_{p,j,K} \left[\nabla_x^{\rho+1} R(c_{p,j,K}, d_{p,j,K}, p) - (-1)^{\rho+1} \nabla_x^{\rho+1} R(c_{p,j,K}, d_{p,j-1,K}, p) \right] \right\}$$

$$\lim_{K \rightarrow \infty} \left\{ K \sum_{p,j} \Delta_{p,j,K}^2 \frac{\sigma^2(c_{p,j,K}, d_{p,j,K}, p) + \sigma^2(c_{p,j,K}, d_{p,j-1,K}, p)}{f_{X|P}(c_{p,j,K}, p) q_p} \right\}$$

Theorem 2 states the rate conditions under which our estimator $\hat{\mu}_{ATE}$ has an asymptotic normal distribution.

Theorem 2. *Assume conditions in 1, 2, 3, 4, 5, 6 hold.*

As $n \rightarrow \infty$, assume that $K \rightarrow \infty$, $h \rightarrow 0$, and $h_I \rightarrow 0$ such that

- $\sqrt{Knh}h^{\rho+1} \rightarrow C \in [0, \infty)$ where ρ is the order of the first step LPR
- $\frac{\sqrt{K} \log n}{\sqrt{nh}} \rightarrow 0$ and $Kh = O(1)$
- $\sqrt{Knh}h_I^{\rho_I+1} \rightarrow 0$ and $1/Kh_I^3 = O(1)$ where ρ_I is the order of the second step multivariate LPR.

then

$$\sqrt{Nh}(\hat{\mu}_{ATE} - \mu_{ATE}) \xrightarrow{d} N(C \cdot AB; AV)$$

where

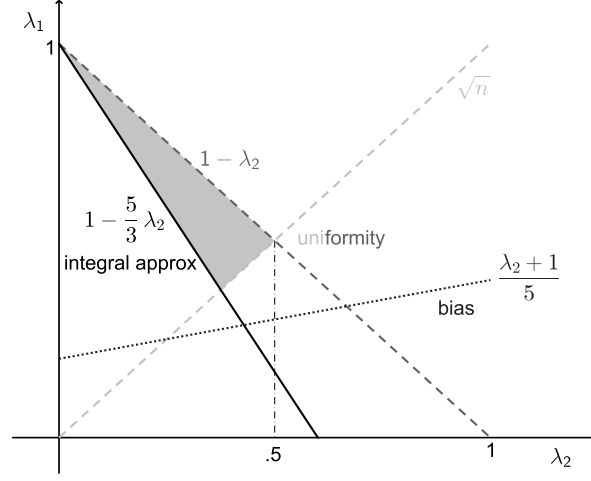
$$AB = \lim_{K, n \rightarrow \infty} \left\{ \sum_{p,j} \Delta_{p,j} \mathcal{B}_{p,j} \right\}$$

$$AV = \lim_{K, n \rightarrow \infty} \left\{ K \sum_{p,j} \Delta_{p,j}^2 \mathcal{V}_{p,j} \right\}$$

and $\mathcal{B}_{p,j}$ and $\mathcal{V}_{p,j}$ are given in equations (8) and (9)

We give a simple example to illustrate the rate conditions. Suppose $h = n^{-\lambda_1}$, $K = n^{\lambda_2}$, $\rho = 1$ (local linear regression in the 1st step), $\rho_I = 3$ (local cubic regression in the 2nd step), and $h_I = MK^{-1/3}$ for some constant M . First, note that $h_I \rightarrow 0$ and $Kh_I^3 = O(1)$. The rate conditions on K and h are illustrated in figure 1 in terms of (λ_1, λ_2) . In this setting, the first set of rate conditions gives $\lambda_1 \geq (\lambda_2 + 1)/(2\rho + 3) = (\lambda_2 + 1)/5$: the bandwidth of the first step estimation has to converge to zero fast enough to control the asymptotic bias (dotted lines); the second set of rate conditions gives $\lambda_1 < 1 - \lambda_2$ and $\lambda_2 \leq \lambda_1$: the total number of cutoffs K cannot grow too fast relatively to the sample size n so to have enough observations around each of the cutoffs to insure the uniformity results (dashed lines). The third rate condition is equivalent to $1 + \lambda_2 (1 - \frac{2}{3}(\rho_I + 1)) < \lambda_1$ or $1 - \lambda_2 (\frac{5}{3}) < \lambda_1$: K has to grow fast enough relatively to n to insure the integral approximation error vanishes faster than the rate of convergence of the estimator $\hat{\mu}_{ATE}$ (solid line). The shaded area in figure 1 below illustrates the set of choices for the bandwidth power λ_1 for a given $\lambda_2 \in (0, .5)$. In this example with $\rho_I = 3$ we do not have asymptotic bias since the shaded area does not touch the bias dotted line. We can use a higher second step polynomial degree of at least $\rho_I = 6$ (expands the shaded area to the left) allowing combinations of λ_1 and λ_2 that lead to asymptotic bias. The maximum convergence rate of the estimator \sqrt{Knh} is equal to \sqrt{n} along the red line.

Figure 1: Rate Conditions of Theorem 2



Notes: Rate conditions of Theorem 2 for an example of sequences $h = n^{-\lambda_1}$, $K = n^{\lambda_2}$, and $h_I = K^{-1/3}$. The first step estimation uses $\rho = 1$ (local linear regression), and the second step estimation uses $\rho_I = 3$ (local cubic regression). The rate conditions are depicted as following: $\sqrt{Knh}h^\rho \rightarrow C$ (dotted line); $\sqrt{K} \log n / \sqrt{nh} \rightarrow 0$ and $Kh = O(1)$ (dashed lines); and $\sqrt{Knh}h_I^{\rho_I} \rightarrow 0$ (solid line).

Another point worth mentioning is that the finite sample bias and standard error expressions are the same for both finite or infinite K asymptotics as long as we use corrected weights. If we define μ_{ALTE} using corrected weights,

$$\mu_{ALTE} = \sum_{p,j} \Delta_{p,j} \beta_{p,j}$$

then the confidence interval for μ_{ALTE} (asymptotics with finite K) has the same formula as the confidence interval for μ_{ATE} (asymptotics with infinite K):

$$\begin{aligned} CI(\mu_{ALTE}; 1 - \alpha) &= CI(\mu_{ATE}; 1 - \alpha) \\ &= \left[\hat{\mu}_{ATE} - h^{\rho+1} \sum_{p,j} \Delta_{p,j} \mathcal{B}_{p,j} \pm z_{\alpha/2} \sqrt{\sum_{p,j} \Delta_{p,j}^2 \mathcal{V}_{p,j}} \right] \end{aligned}$$

where $z_{\alpha/2}$ is the $1 - \alpha/2$ percentile of the standard normal distribution. This confidence

interval has asymptotic coverage of $1 - \alpha$ under both types of asymptotics (finite or infinite K).

The asymptotic bias and variance terms of theorem 2 can be consistently estimated using the procedures discussed at the end of section 2. One has to compute $\widehat{\mathcal{B}}_{p,j}$ and $\widehat{\mathcal{V}}_{p,j}$ and the weighted sum using the corrected weights $\Delta_{p,j}$. Sufficient moment and rate conditions give consistency of these estimators under the asymptotics with large number of cutoffs.

4 Parametric Heterogeneity

In the third case of heterogeneity assumptions, the researcher specifies a parametric functional form for the treatment effect function $\beta(\cdot)$. Economic theory or a priori knowledge guides the choice of a functional form that credibly summarizes the heterogeneity of treatment effects. For example, Lazear (2001) presents a well known formalization of a theory of educational output as a function of class size, teacher quality and student characteristics. In this section we discuss both the sharp and fuzzy RDD cases in separate subsections. For each case, we give sufficient conditions for identification and asymptotically normal estimation of the functional form parameters. The ATE is simply a linear combination of these parameters. Different than section 3, identification does not require a large number of cutoffs because of the parametric functional form assumption. The default asymptotic exercise has the sample size growing to infinity but the number of cutoffs fixed. In the sharp case, we show that our asymptotic normality result holds even when the number of cutoffs goes to infinity. This is not pursued in the fuzzy case, because the definition of the different compliance behaviors depends on the number of cutoffs being finite.

4.1 Sharp RDD

Thus far we have been able to make inferences on the ATE over general distributions of individuals when the treatment effect function $\beta(\cdot)$ is an unknown ‘infinite’ dimensional

object. In this section, economic theory or a priori knowledge of the researcher can be used to restrict such a function to be an unknown ‘finite’ dimensional object. Besides the ATE, researchers are also interested in learning the form of $\beta(\cdot)$ and how treatment effects vary with changes in (x, d, d') . In our high school assignment example, we may be interested in learning whether the average return to school quality varies with the test score which is a measure of ability. If $\beta(x, d, d') = \theta_1(d' - d) + \theta_2(d' - d)x + \theta_3(d' - d)x^2$ for unknown θ , we can test the hypothesis that $\theta_3 = 0$. Therefore, a second type of summary measure for heterogeneous treatment effects is the vector of parameters θ of this functional form. The parametric functional form assumption is formally stated below.

Assumption 7. *For a finite vector of real valued functions $W(x, d, d') = [W_1(x, d, d'), \dots, W_q(x, d, d')]'$ known to the researcher, there exists a unique $\theta_0 \in \mathbb{R}^q$ such that*

$$\beta(x, d, d') \equiv W(x, d, d')'\theta \iff \theta = \theta_0$$

Moreover, for every x, d_1, d_2, d_3 , $W(x, d_1, d_3) = W(x, d_1, d_2) + W(x, d_2, d_3)$.⁹ For a weighing density $\omega(x, d, d')$ chosen by the researcher, $\omega(x, d, d')W(x, d, d')$ is a Riemann integrable function over the set \mathcal{C} .

It is worth mentioning that assumption 7 is weaker than the common practice in applied work to specify a parametric functional form on the conditional mean function of the outcome Y . A parametric assumption on $\beta(c, d, d')$ is equivalent to a semi-parametric assumption on the conditional mean of $E[Y_i(d)|X_i = c, P_i = p]$. To see this, fix a baseline treatment dose $d_0 \in \mathcal{D}$. For any $(c, d, p) \in \mathcal{X} \times \mathcal{D} \times \mathcal{P}$,

⁹This summability condition is assumed here for consistency with the linearity of the expectation operator. In other words, the treatment effect function $\beta(\cdot)$ is defined in equation (2) using the expectation operator on the distribution of the change in potential outcomes. Therefore, $\beta(x, d_1, d_3) = \beta(x, d_1, d_2) + \beta(x, d_2, d_3)$, and the parametric functional form needs to be consistent with this. We work with parametric functional forms that are linear in θ to make identification in fuzzy case more tractable. Results in the sharp case also apply to functional forms that are non-linear but differentiable in θ .

$$E[Y_i(d)|X_i = c, P_i = p] = \beta(c, d_0, d) + E[Y_i(d_0)|X_i = c, P_i = p]$$

Under assumption 7, if you know θ_0 , you know the entire function $\beta(c, d, d')$, but you still need knowledge of $E[Y_i(d_0)|X_i = c, P_i = p]$ as a function of (c, p) in order to retrieve the entire function $E[Y_i(d)|X_i = c, P_i = p]$ for all (c, d, p) . In other words, our functional form restriction is robust to misspecification of $E[Y_i(d_0)|X_i = c, P_i = p]$. Robustness to misspecification in the conditional mean of Y is an useful property because empirical evidence suggests the conditional mean of Y to be a much more complex function than the treatment effect function $\beta(\cdot)$. In this case, misspecifying the conditional mean of Y leads to a larger bias than misspecifying the treatment effect function $\beta(x, d, d')$.

The ATE in the third case of heterogeneity assumptions is simply the integral of the functional form of assumption 7 with a weighting density $\omega(x, d, d')$ chosen by the researcher.¹⁰

$$\begin{aligned} \mu_{ATE} &= \int_{\mathcal{C}} \omega(x, d, d') \beta(x, d, d') d(x, d, d') \\ &= \left(\int_{\mathcal{C}} \omega(x, d, d') W(x, d, d') d(x, d, d') \right) \theta_0 \equiv Z \theta_0 \end{aligned} \tag{13}$$

where Z is a known $1 \times q$ vector that can be computed once the researcher specifies \mathcal{C} , $W(x, d, d')$ and $\omega(x, d, d')$.

Lemma 3 shows that θ_0 is identified as long as there is sufficient variation in cutoff characteristics relative to the basis functions $W(x, d, d')$. The ATE μ_{ATE} is also identified because it is a known linear function of θ_0 .

Lemma 3. *Assume $\beta(c_{p,j}, d_{p,j-1}, d_{p,j}, p)$ is identified for every $p \in \mathcal{P}$ and $j \in \mathcal{J}_p$, and assumptions 1 and 7 hold. Let $W_{p,j} = W(c_{p,j}, d_{p,j-1}, d_{p,j})$, where $W(\cdot)$ is the vector basis*

¹⁰The integral is computed over set \mathcal{C} by default, but it could be any other subset of \mathbb{R}^3 in which the researcher believes the function form credibly explains the heterogeneity of treatment effects.

function of assumption 7. If $\sum_{p,j} W_{p,j} W'_{p,j}$ is invertible, then θ_0 is identified and

$$\theta_0 = \left(\sum_{p,j} W_{p,j} W'_{p,j} \right)^{-1} \sum_{p,j} W_{p,j} B_{p,j}$$

where $B_{p,j}$ is defined in equation 3

The estimation of θ_0 is laid out in two steps. The first step is exactly the same as in the previous sections. We use observations near each of the cutoffs in each sub-population to estimate $B_{p,j}$ non-parametrically by LPR (equations 4, 5 and 6). In the second stage, we regress $\hat{B}_{p,j}$ on the basis functions evaluated at each cutoff-dose explanatory variables $W_{p,j} = W(c_{p,j}, d_{p,j-1}, d_{p,j})$ for $W(\cdot)$ of assumption 7 to obtain $\hat{\theta}$. Since the treatment effect function is parametric, we can weight first stage estimates differently to minimize the mean squared error (MSE) of $\hat{\theta}$. More specifically, we stack all $\hat{B}_{p,j}$ into a $K \times 1$ vector \hat{B} , and stack all $W_{p,j}$ into a $K \times q$ matrix W . Using a $K \times K$ symmetric and positive definite weighting matrix Ω chosen by the researcher, $\hat{\theta}$ is the solution to the following weighted least squares problem:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \left(\hat{B} - W\theta \right)' \Omega \left(\hat{B} - W\theta \right)$$

As in equation 13, the estimator for μ_{ATE} is a linear combination of $\hat{\theta}$.

$$\hat{\mu}_{ATE} = Z\hat{\theta}$$

For a fixed number of cutoffs, as the sample size n increases and the bandwidth h converges to zero, each individual observation is used only once in the whole estimation after a large n . The estimated treatment effects are independent of each other across different cutoffs. The asymptotic distribution of each element of $\hat{\theta}$ is a linear combination of the

asymptotic normal distribution of each $\hat{B}_{p,j}$ (Lemma 7 in the Appendix)

Theorem 3. *Suppose assumptions 1, 2, 3, 4, 7 hold. As $n \rightarrow \infty$ and $h \rightarrow 0$, assume $nh \rightarrow \infty$ and $\sqrt{nh}h^{\rho+1} \rightarrow C \in [0, \infty)$. Then,*

$$\begin{aligned} \sqrt{nh} (\hat{\theta} - \theta_0) &\xrightarrow{d} N(C(W'\Omega W)^{-1}W'\Omega\mathcal{B}; (W'\Omega W)^{-1}W'\Omega\mathcal{V}\Omega W(W'\Omega W)^{-1}) \\ \sqrt{nh} (\hat{\mu}_{ATE} - \mu_{ATE}) &\xrightarrow{d} N(ZC(W'\Omega W)^{-1}W'\Omega\mathcal{B}; Z(W'\Omega W)^{-1}W'\Omega\mathcal{V}\Omega W(W'\Omega W)^{-1}Z') \end{aligned}$$

Moreover, the asymptotic MSE of either $\sqrt{nh} (\hat{\theta} - \theta_0)$ or $\sqrt{nh} (\hat{\mu}_{ATE} - \mu_{ATE})$ is minimized when $\Omega = (C^2\mathcal{B}\mathcal{B}' + \mathcal{V})^{-1}$. Below, the definitions used:

$$\mathcal{B} = [\mathcal{B}_{1,K(1)}, \dots, \mathcal{B}_{K(1),K(1)}, \mathcal{B}_{1,K(2)}, \dots, \mathcal{B}_{K(P),K(P)}]'$$

the formula for $\mathcal{B}_{p,j}$ is given in equation 8

$$\mathcal{V} = \text{diag} \{ \mathcal{V}_{1,K(1)}, \dots, \mathcal{V}_{K(1),K(1)}, \mathcal{V}_{1,K(2)}, \dots, \mathcal{V}_{K(P),K(P)} \}'$$

the formula for $\mathcal{V}_{p,j}$ is given in equation 9

$$Z = \left(\int_{\mathcal{C}} \omega(x, d, d') W(x, d, d') d(x, d, d') \right)$$

$$W = [W'_{1,K(1)}, \dots, W'_{K(1),K(1)}, W'_{1,K(2)}, \dots, W'_{K(P),K(P)}]'$$

$$W_{p,j} = W(c_{p,j}, d_{p,j-1}, d_{p,j})$$

The estimator for the asymptotic variance and bias are straightforward. We know W , Z , and we need to obtain \hat{C} , $\hat{\mathcal{B}}$, and $\hat{\mathcal{V}}$. We obtain estimates for $\mathcal{B}_{p,j}$ and $\mathcal{V}_{p,j}$ according to the procedure discussed in the end of section 2. Once we have $\hat{\mathcal{B}}$ and $\hat{\mathcal{V}}$, we can compute the optimal weighting matrix $\hat{\Omega} = (\hat{C}^2\hat{\mathcal{B}}\hat{\mathcal{B}}' + \hat{\mathcal{V}})^{-1}$, where $\hat{C} = \sqrt{nh}h^{\rho+1}$.

The default asymptotic exercise for this section has the sample size growing large but the number of cutoffs fixed. In the third case of heterogeneity assumptions, the treatment function is an unknown object of only finite dimension, and we do not need the number of

cutoffs to grow to infinity to approximate the integral average treatment effect. Nevertheless, under conditions similar to theorem 2, we also obtain asymptotic normality of $\hat{\theta}$ under the asymptotics with a large number of cutoffs.

Corollary 1. *Assume conditions in 1, 2, 3, 4, 5(a,b), 6(a,c,d,e) hold. Assume 7 holds and its vector valued function $W(x, d, d')$ is bounded. Assume 6(f) holds for the $q \times 1$ vector valued weights*

$$w_{p,j} = (W' \Omega W)^{-1} W' \Omega_{\bullet, (p,j)}$$

in the place of $\Delta_{p,j}$, where $\Omega_{\bullet, (p,j)}$ is the column of Ω associated with cutoff (p, j) , and that $\max_{p,j} \|(W' \Omega W)^{-1} W' \Omega_{\bullet, (p,j)}\| = O(K^{-1})$. As $n \rightarrow \infty$, assume $K \rightarrow \infty$, and $h \rightarrow 0$ such that

- $\frac{\sqrt{K} \log n}{\sqrt{nh}} \rightarrow 0$ and $Kh = O(1)$
- $\sqrt{Knh} h^{\rho+1} \rightarrow C \in [0, \infty)$ where ρ is the order of the LPRs

then

$$\begin{aligned} \sqrt{Knh} (\hat{\theta} - \theta_0) &\xrightarrow{d} N(C \cdot AB_{\theta}; AV_{\theta}) \\ \sqrt{Knh} (\hat{\mu}_{ATE} - \mu_{ATE}) &\xrightarrow{d} N(Z' C' AB_{\theta}; Z' AV_{\theta} Z') \end{aligned}$$

where

$$\begin{aligned} AB_{\theta} &= \lim_{K, n \rightarrow \infty} \sum_{p,j} w_{p,j} \mathcal{B}_{p,j} \\ AV_{\theta} &= \lim_{K, n \rightarrow \infty} K \sum_{p,j} w_{p,j} w'_{p,j} \mathcal{V}_{p,j} \end{aligned}$$

where $\mathcal{B}_{p,j}$ and $\mathcal{V}_{p,j}$ are given in equations 8 and 9.

Both estimators $\widehat{\theta}$ and $\widehat{\mu}_{ATE}$ have a faster convergence rate under the large K asymptotics. Differently than theorem 2, there is no lower bound requirement on the speed that K grows relatively to n which includes the fixed K case. Comparing the finite sample expressions for bias and variance that one obtains from theorem 3 or theorem 1, we notice that they do not change whether K grows large or not. Consistent estimators for the asymptotic bias and variance terms are constructed using the estimators $\widehat{\mathcal{B}}_{p,j}$ and $\widehat{\mathcal{V}}_{p,j}$ proposed in the end of section 2. Sufficient moment and rate conditions give consistency of these estimators under the asymptotics with large number of cutoffs.

4.2 Fuzzy RDD

Another key advantage of the third case of heterogeneity assumptions is that a parametric functional form obtains identification in the fuzzy RDD case with multiple cutoffs. In the fuzzy RDD case, a “sharp RDD like” treatment eligibility schedule $D(x, p)$ (defined in eq. 1) applies to most individuals, but the rest of individuals deviates from such treatment schedule for unobserved reasons. In the high school assignment example, students may choose to go to a school that is not the best school they get in. For instance, a student may want to attend the same high school as does a certain friend or sibling.¹¹ Another example is Garibaldi, Giavazzi, Ichino, and Rettore (2012), where the schedule of tuition subsidies applies to most students in Bocconi University, but the university reserves the right to grant certain students different subsidies after reassessing their ability to pay.

The fuzzy RDD case is modeled in terms of the potential treatment assignment framework. Let $(\Omega, \mathcal{A}, \mathbb{P})$ denote a probability space for the population of interest. For each individual $\omega \in \Omega$, we define the potential treatment assignments for each treatment eli-

¹¹The RDD assignment may be fuzzy for application-specific reasons. One example is the case where the assignment of individuals into different treatments is made through a matching mechanism, and the econometrician does not observe all the individual characteristics used in the matching algorithm. This is the reason why the RDD in PEU is fuzzy: based on the entire distribution of test scores and preferences, the central planner ranks students by their test scores and assigns each one of them to her most preferred school among those schools with vacancies. We keep the simple example of the high school assignment problem in the main text of ease of exposition.

gibility in sub-population $p \in \mathcal{P}$ by the measurable function $\mathcal{I}_p : \Omega \times \mathcal{J}_p^0 \rightarrow \mathcal{J}_p^0$, where $\mathcal{J}_p = \{1, \dots, K(p)\}$ and $\mathcal{J}_p^0 = \mathcal{J}_p \cup \{0\}$. That is, $\mathcal{I}_p(\omega, j)$ denotes the treatment individual ω receives given she is eligible to receive treatment $j \in \mathcal{J}_p^0$ and is in sub-population $p \in \mathcal{P}$. We do not observe the potential treatment assignments but just the actual treatment dose received $D_i : \Omega \rightarrow \mathcal{D}$.

$$D_i(\omega) = \sum_{\substack{p \in \mathcal{P}, j \in \mathcal{J}_p \\ l \in \mathcal{J}_p^0}} \mathbb{I}\{c_{p,j} \leq X_i(\omega) < c_{j+1,p}\} \mathbb{I}\{P_i(\omega) = p\} \mathbb{I}\{\mathcal{I}_p(\omega, j) = l\} d_{p,l}$$

Treatment doses $d_{p,j}$ are assumed to be increasing in $j \in \mathcal{J}_p^0$ for every $p \in \mathcal{P}$, but this restriction can be relaxed (see formal definition of compliance groups below and footnote).

We build on classical definitions of compliance behaviors (e.g. Imbens and Rubin (1997)) and define three types of compliance groups in a sub-population p with multiple treatments. We use a simple example with 3 schools and one sub-population to introduce the different compliance behaviors. That is, $P = 1$, $K = 2$, $\mathcal{J}_1^0 = \{0, 1, 2\}$, and we assume $d_{1,0} < d_{1,1} < d_{1,2}$. Table 1 lists all possible treatment eligibility and assignment combinations.

Table 1: Different Compliance Behaviors

$\mathcal{I}_1(0)$	$\mathcal{I}_1(1)$	$\mathcal{I}_1(2)$	Type
0	0	0	never-changers
1	1	1	
2	2	2	
0	0	2	compliers
0	1	1	
0	1	2	
1	1	2	
0	0	1	defiers
0	1	0	
0	2	0	
0	2	1	
0	2	2	
1	0	0	
1	0	1	
1	0	2	

$\mathcal{I}_1(0)$	$\mathcal{I}_1(1)$	$\mathcal{I}_1(2)$	Type
1	0	2	defiers
1	1	0	
1	2	0	
1	2	1	
1	2	2	
2	0	0	
2	0	1	
2	0	2	
2	1	0	
2	1	1	
2	1	2	
2	2	0	
2	2	1	

Notes: All possible realizations of the random function $\mathcal{I}_1(j)$ which denotes the treatment dose received given eligibility for treatment dose $j \in \mathcal{J}_1^0$.

The three different behaviors are defined in terms of changes in treatment eligibility. ‘Never-changers’ are those whose treatment received never changes when eligibility changes. The treatment received by ‘compliers’ or ‘defiers’ changes at least once when eligibility changes. ‘Compliers’ are those whose treatment received changes if and only if it changes to the better treatment dose they become eligible for. ‘Defiers’ change to a treatment dose different from one they are eligible for. In the case of two schools, our definition is equivalent to the classical definition of compliers and defiers.

In the high school assignment case, an example of a ‘never-changer’ is a student who strongly prefers the high school with the lowest admission cutoff and will attend that high school even if she is admitted to better schools. An example of a ‘complier’ is a student who attends the best school into which she is admitted or a student who chooses the best school among those nearby schools. If a student has rational preferences, is never indifferent, and can always pick a high school among those schools with admission cutoffs that are less or

equal than her test score, then she is never a ‘defier’: as her test score increases, a new school is added to her choice-set of schools; she either chooses to go to the new school to which she becomes eligible for, or she stays at the school which she preferred the most prior to the increase in her choice-set.

These three groups are measurable sets that form a different partition of Ω for each $p \in \mathcal{P}$. They are formally defined below.¹²

$$\begin{aligned}\mathbf{G}_{nc,p} &= \{\omega \in \Omega : \forall j \in \mathcal{J}_p, \mathcal{I}_p(\omega, j) = \mathcal{I}_p(\omega, j-1)\} \\ \mathbf{G}_{c,p} &= \{\omega \in \Omega : \forall j \in \mathcal{J}_p \text{ st } \mathcal{I}_p(\omega, j-1) \neq \mathcal{I}_p(\omega, j) \text{ we have } \mathcal{I}_p(\omega, j-1) < \mathcal{I}_p(\omega, j) = j\} \\ \mathbf{G}_{d,p} &= \{\omega \in \Omega : \exists j \in \mathcal{J}_p \text{ st } \mathcal{I}_p(\omega, j-1) > \mathcal{I}_p(\omega, j) \text{ or } \mathcal{I}_p(\omega, j-1) < \mathcal{I}_p(\omega, j) \neq j\}\end{aligned}$$

Our definition of the compliance groups is suited to a finite number of cutoffs, and a population with an infinite number of cutoffs would require a more complex definition of compliance. It is plausible to assume that the treatment received never changes unless it changes to comply with the change in treatment eligibility. This minimal assumption rules out defiers in the population. For never-changers, we can never identify treatment effects because they never undergo a change in treatment dose. Because of the multiplicity of treatments, compliers can differ from each other when it comes to the number of treatments they comply with. For example, the student who is willing to attend the best school possible complies with all changes in treatment eligibility. On the other hand, the student who is willing to attend the best school possible within a certain distance from home only complies with some of the changes in treatment eligibility. We do not observe potential treatment assignments but only the treatments individuals actually receive. Therefore, we cannot

¹²The definitions of the three groups is based on the treatment doses being increasing in j for each sub-population, $d_{p,0} < d_{p,1} < \dots < d_{p,K(p)}$. The definitions can be changed to accommodate decreasing or non-monotonic treatment doses. For example, if a school with a higher admission cutoff happens to have a lower quality, or in the class-size rule applications when the class size drops after each cutoff. Compliers could then be defined as those who comply at least once to some eligibility change no matter what dose they receive prior to the change.

distinguish one type of complier from another. For a sub-population $p \in \mathcal{P}$, treatment doses $d, d' \in \mathcal{D}$, we define the treatment effect on compliers to be given by the same function of (x, d, d', p) across different types of compliers.

$$\beta_c(x, d, d', p) = E[Y_i(d') - Y_i(d) | X_i = x, P_i = p, G] \quad \forall G \in \mathcal{A} : G \subseteq \mathbf{G}_{c,p}$$

Assumption 8 is a generalized version of the sufficient conditions for identification of the treatment effect on compliers in the case with one cutoff (Hahn, Todd, and Van der Klaauw (2001)).

Assumption 8.

- (i) *There are no defiers: $\mathbb{P}[\mathbf{G}_{d,p}] = 0$ for $\forall p \in \mathcal{P}$*
- (ii) *For any $p \in \mathcal{P}$, $d \in \mathcal{D}$, and $\forall G \in \mathcal{A} : G \subseteq \mathbf{G}_{c,p}$, $E[Y_i(d) | X_i = x, P_i = p, G]$ is a continuous function of x*
- (iii) *There exists $\bar{e} > 0$ small such that for any $p \in \mathcal{P}$, $j \in \mathcal{J}_p$, $l \in \mathcal{J}_p^0$,*

$$(\mathcal{I}_p(j), \mathcal{I}_p(j-1), Y_i(d_{p,l})) \perp X_i \mid X_i \in [c_{p,j} \pm \bar{e}], P_i = p$$
where $\mathcal{I}_p(\cdot)$ is the random variable defined by the function $\mathcal{I}_p(\omega, \cdot)$.

In the case of table 1, after we rule out defiers, we are left with two cutoffs and 3 possible treatment effects on compliers: $\beta_c(c_{1,1}, d_{1,0}, d_{1,1}, 1)$ in cutoff $c_{1,1}$; $\beta_c(c_{1,2}, d_{1,0}, d_{1,2}, 1)$ and $\beta_c(c_{1,2}, d_{1,1}, d_{1,2}, 1)$. Comparing the average of individuals around cutoff $c_{1,1}$ identifies treatment effect $\beta_c(c_{1,1}, d_{1,0}, d_{1,1}, 1)$. Identification is not possible in cutoff $c_{1,2}$ because there are two treatment effects to be retrieved from the observed change in average outcome around such cutoff. It becomes necessary to impose some assumption on how the identified treatment effect in cutoff $c_{1,1}$ relates to the unidentified effects in cutoff $c_{2,1}$. We use the parametric functional form specified by the researcher in the third case of heterogeneity assumptions to obtain identification in the fuzzy case.

Similarly to the sharp case, assumption 1 applied to β_c allows one to pool observations from many sub-populations. Lemma 4 shows that the observed change in average outcome at a given cutoff is a weighted average of treatment effects on compliers who switch from various doses into the dose associated with that cutoff. If the treatment effect function is linear in a vector of parameters, then the observed jump in outcomes is also linear in those parameters for which we can solve.

Lemma 4. *Under assumption 8,*

$$B_{p,j} = \sum_{\substack{l \in \mathcal{J}_p^0 \\ l < j}} \omega_{p,j,l} \beta_c(c_{p,j}, d_{p,l}, d_{p,j}, p)$$

where

$$\omega_{p,j,l} = \lim_{e \downarrow 0} \{ \mathbb{P}[D_i = d_{p,l} | X_i = c_{p,j} - e, P_i = p] - \mathbb{P}[D_i = d_{p,l} | X_i = c_{p,j} + e, P_i = p] \} \quad (14)$$

and $B_{p,j}$ is defined in equation 3

Moreover, suppose assumptions 1 and 7 hold for β_c . Define $\widetilde{W}_{p,j} = \sum_{l \in \mathcal{J}_p^0, l < j} \omega_{p,j,l} W_{p,j,l}$ with $W_{p,j,l} = W(c_{p,j}, d_{p,l}, d_{p,j})$ for the vector basis function $W(\cdot)$ of assumption 7. If $\sum_{p,j} \widetilde{W}_{p,j} \widetilde{W}_{p,j}'$ is invertible, then θ_0^c is identified and

$$\theta_0^c = \left(\sum_{p,j} \widetilde{W}_{p,j} \widetilde{W}_{p,j}' \right)^{-1} \sum_{p,j} \widetilde{W}_{p,j} B_{p,j}$$

The ATE is simply the integral of the functional form β_c over the relevant set \mathcal{C} where the researcher chooses the weighting density $\omega(x, d, d')$.

$$\begin{aligned}
\mu_{ATE}^c &= \int_{\mathcal{C}} \omega(x, d, d') \beta_c(x, d, d') d(x, d, d') \\
&= \left(\int_{\mathcal{C}} \omega(x, d, d') W(x, d, d') d(x, d, d') \right) \theta_0^c \equiv Z \theta_0^c
\end{aligned} \tag{15}$$

Lemma 4 suggests a two-step estimation procedure for θ_0^c where first-step estimates of $B_{p,j}$ are regressed on estimates of $\widetilde{W}_{p,j}$. The first-step is computationally similar to the previous sections. For a sub-population p , cutoff j , we estimate the jump $B_{p,j}$ in the conditional mean of Y using observations around cutoff (p, j) (equations 4, 5, and 6). The novelty of the first-step is the estimation of the jump in the probability of receiving a given treatment dose in a given cutoff. For any $p \in \mathcal{P}$, $j \in \mathcal{J}_p$, $l \in \mathcal{J}_p^0, l < j$, we use LPRs to estimate $\omega_{p,j,l}$ defined in equation 14.

Similarly to the procedures described in equations 4 - 6, we estimate $\omega_{p,j,l}$ by regressing $\mathbb{I}\{D_i = d_{p,l}\}$ on polynomials of X on each side of the cutoff $c_{p,j}$. We choose a bandwidth $h_\omega > 0$ and the order of the polynomial ρ_ω .

$$\widehat{\omega}_{p,j,l} = \widehat{a}_{p,j}^- - \widehat{a}_{p,j}^+ \tag{16}$$

$$(\widehat{a}_{p,j}^+, \widehat{b}_{p,j}^+) = \underset{(a,b)}{\operatorname{argmin}} \sum_{i=1}^n k\left(\frac{X_i - c_{p,j}}{h_\omega}\right) v_i^{p,j+} \left[\mathbb{I}\{D_i = d_{p,l}\} - a - b_1(X_i - c_{p,j}) - \dots - b_{\rho_\omega}(X_i - c_{p,j})^{\rho_\omega} \right]^2 \tag{17}$$

$$(\widehat{a}_{p,j}^-, \widehat{b}_{p,j}^-) = \underset{(a,b)}{\operatorname{argmin}} \sum_{i=1}^n k\left(\frac{X_i - c_{p,j}}{h_\omega}\right) v_i^{p,j-} \left[\mathbb{I}\{D_i = d_{p,l}\} - a - b_1(X_i - c_{p,j}) - \dots - b_{\rho_\omega}(X_i - c_{p,j})^{\rho_\omega} \right]^2 \tag{18}$$

where $v_i^{p,j+} = \mathbb{I}\{c_{p,j} \leq X_i < c_{p,j} + h_\omega, P_i = p\}$, $v_i^{p,j-} = \mathbb{I}\{c_{p,j} - h_\omega < X_i < c_{p,j}, P_i = p\}$. We compute $\widehat{\omega}_{p,j,l}$ for every $p \in \mathcal{P}$, $j \in \mathcal{J}_p$, $l \in \mathcal{J}_p^0, l < j$.

Using these estimated probabilities $\widehat{\omega}_{p,j,l}$, we compute an estimate $\widehat{\widetilde{W}}_{p,j}$ for the weighted

average $\widehat{W}_{p,j}$ of basis functions evaluated at the explanatory variables of each change in dose.

$$\widehat{W}_{p,j} = \sum_{l \in \mathcal{J}_p^0, l < j}^{K(p)} \widehat{\omega}_{p,j,l} W(c_{p,j}, d_{p,l}, d_{p,j}) = \sum_{l \in \mathcal{J}_p^0, l < j} \widehat{\omega}_{p,j,l} W_{p,j,l} \quad (19)$$

for the $q \times 1$ vector of basis functions $W(\cdot)$ of assumption 7.

The regression of $\hat{B}_{p,j}$ on $\widehat{W}_{p,j}$ gives an estimate for θ_0 . More specifically, we stack all $1 \times q$ vectors $\widehat{W}_{p,j}^\top$ into the $K \times q$ matrix \widehat{W} , and $\hat{B}_{p,j}$ into the $K \times 1$ vector \hat{B} . Given a $K \times K$ symmetric and positive definite weighting matrix Ω , our estimator for θ_0^c is a solution to the following weighted least squares problem:

$$\hat{\theta}^c = \underset{\theta}{\operatorname{argmin}} \left(\hat{B} - \widehat{W}\theta \right)' \Omega \left(\hat{B} - \widehat{W}\theta \right) \quad (20)$$

Following equation 15, the estimator for μ_{ATE}^c is a linear combination of $\hat{\theta}^c$.

$$\widehat{\mu}_{ATE}^c = Z\hat{\theta}^c$$

where $Z = \left(\int_{\mathcal{C}} \omega(x, d, d') W(x, d, d') d(x, d, d') \right)$ is known.

We state sufficient conditions for asymptotic normality of these estimators in the default asymptotics with large number of observations but fixed number of cutoffs. The asymptotics with a large number of cutoffs is not pursued because of tractability. Our definitions of compliance groups relies on the number of cutoffs being finite. Moreover, a large number of cutoffs increases the number of $\hat{\omega}$ to be estimated in the first stage. Next, we extend the smoothness assumptions on the conditional mean of Y_i (assumption 4) to the conditional probabilities of potential treatment doses.

Assumption 9. *For every $p \in \mathcal{P}$, $j \in \mathcal{J}_p$, $l \in \mathcal{J}_p^0$*

- $R_Y(x, j, l, p) = E[Y_i(d_{p,l}) | X_i = x, P_i = p, \mathcal{I}_p(j) = l]$ is $\rho + 1$ times continuously differ-

entiable w.r.t. x with $\rho + 1$ -th partial derivative $\nabla_x^{\rho+1} R_Y(x, j, l, p)$

- $R_{Y^2}(x, j, l, p) = E[Y_i(d)^2 | X_i = x, P_i = p, \mathcal{I}_p(j) = l]$ is a continuous function of x
 - $\widetilde{W}^\top \widetilde{W} = \sum_{p,j} \widetilde{W}_{p,j} \widetilde{W}'_{p,j}$ is invertible
 - Define $\bar{\rho} = \max\{\rho, \rho_\omega\}$, where ρ is the order of the LPR in equations 5 and 6, and ρ_ω is the order of the LPR in equations 17 and 18.
- $R_D(x, j, l, p) = \mathbb{P}[\mathcal{I}_p(j) = l | X_i = x, P_i = p]$ is $\bar{\rho} + 1$ times continuously differentiable w.r.t. x with $\bar{\rho} + 1$ -th partial derivative $\nabla_x^{\bar{\rho}+1} R_D(x, j, l, p)$.

Theorem 4. Suppose assumptions 1 and 7 hold for the treatment effect on compliers function $\beta_c(c, d, d')$. Suppose assumptions 2, 3, 8, 9 hold.

As $n \rightarrow \infty$, $h \rightarrow 0$, and $h_\omega \rightarrow 0$, assume

- $nh \rightarrow \infty$, $nh_\omega \rightarrow \infty$
- $h/h_\omega = o(1)$
- $\sqrt{nh} h^{\rho+1} \rightarrow C \in [0, \infty)$, $\sqrt{nh_\omega} h_\omega^{\rho_\omega+1} = O(1)$

then

$$\begin{aligned} \sqrt{nh} (\hat{\theta}^c - \theta_0^c) &\xrightarrow{d} N \left(C (\widetilde{W}' \Omega \widetilde{W})^{-1} \widetilde{W}' \Omega \mathcal{B}, (\widetilde{W}' \Omega \widetilde{W})^{-1} \widetilde{W}' \Omega \mathcal{V} \Omega \widetilde{W} (\widetilde{W}' \Omega \widetilde{W})^{-1} \right) \\ \sqrt{nh} (\hat{\mu}_{ATE}^c - \mu_{ATE}^c) &\xrightarrow{d} N \left(Z C (\widetilde{W}' \Omega \widetilde{W})^{-1} \widetilde{W}' \Omega \mathcal{B}, Z (\widetilde{W}' \Omega \widetilde{W})^{-1} \widetilde{W}' \Omega \mathcal{V} \Omega \widetilde{W} (\widetilde{W}' \Omega \widetilde{W})^{-1} Z' \right) \end{aligned}$$

Moreover, the asymptotic MSE of either $\sqrt{nh} (\hat{\theta}^c - \theta_0^c)$ or $\sqrt{nh} (\hat{\mu}_{ATE}^c - \mu_{ATE}^c)$ is minimized when $\Omega = (C^2 \mathcal{B} \mathcal{B}' + \mathcal{V})^{-1}$. Below, the definitions used:

$$\mathcal{B} = [\mathcal{B}_{1,K(1)}, \dots, \mathcal{B}_{K(1),K(1)}, \mathcal{B}_{1,K(2)}, \dots, \mathcal{B}_{K(P),K(P)}]'$$

the formula for $\mathcal{B}_{p,j}$ is given in equation 8

$$\mathcal{V} = \text{diag} \{ \mathcal{V}_{1,K(1)}, \dots, \mathcal{V}_{K(1),K(1)}, \mathcal{V}_{1,K(2)}, \dots, \mathcal{V}_{K(P),K(P)} \}'$$

the formula for $\mathcal{V}_{p,j}$ is given in equation 9

Estimates for the asymptotic bias and variance as well as the optimal weighting matrix are obtained in exactly the same way as proposed in section 4.1, theorem 3.

5 Application

In this section, we illustrate our methods using the data from PEU on the high school assignment in Romania.¹³ PEU contributes to literature by providing rigorous evidence of the impacts of going to a better school on the academic performance of students and on the behavior of parents and teachers. To our knowledge, they were the first ones to apply RDD to a dataset with variation in cutoff-dose values much larger than most RDD applications. As rich data become available, applications of RDD with many thresholds will become even more common reinforcing the already existing demand for our methods. Our purpose in this section is to show the empirical relevance of three of our main contributions:

- (i) the interpretation of the average effect depends on the weighting scheme implied by the researcher's policy question because local treatment effects are heterogeneous; we give one example of policy question where the average effect obtained by normalizing all cutoffs to zero does not provide the right answer;
- (ii) some policy questions target an ATE rather than an ALTE; we give one example of

¹³The dataset is available online through the website of the *American Economic Review* where PEU was published.

such policy question and show that the estimated ALTE provides a different answer than the estimated ATE;

- (iii) a parametric functional form yields (i) an optimal weighting scheme that increases estimation precision; (ii) identification of treatment effects for the sub-group of ‘compliers’ (fuzzy RDD case); we show that the optimal weighting scheme changes the inference conclusions, and that the effect on ‘compliers’ is much larger than the Intent-to-Treat (ITT) effect which is obtained when the sharp RDD estimator is used in a fuzzy RDD application

The administrative data from Romania covers 3 cohorts of 9 grade students for the years of 2001, 2002 and 2003. The size of the cohorts are 107812, 110912, and 115413, with a total of 334137 observations. We only describe the essential elements of the high school assignment mechanism and refer the reader to PEU for all the details. The assignment to high school is nationally centralized by the Ministry of Education. At the end of grade 8, students submit a transition score and a complete ranking of preferences for high schools. The transition score is an average of the student’s performance in a national eighth grade exam and the student’s grade point average of grades 5-8. The Ministry of Education ranks students by their transition score and no other criteria. The mechanism assigns the student in the first place to her most preferred school, the student in the second place to her most preferred school, etc. Each school has a fixed number of vacancies, so the mechanism eventually reaches a student in the rank whose most preferred school is full: it assigns this student to her most preferred school among those with vacancies. It is assumed that students always prefer the high schools in their home town to high schools in other towns which is reasonably the case for 13-14 year old kids living with their parents. Students cannot decline their assignment and have incentives to truthfully reveal their preference rankings.

Another contribution of PEU is to analyze the data produced by such assignment mechanism using RDD methods. We observe the year, the town, the transition score X and the school each student is assigned to. The summary measure of school quality (treatment dose

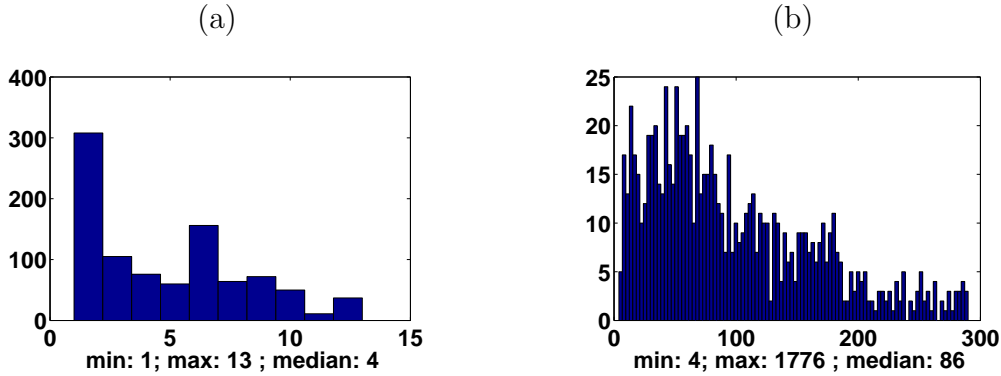
d) is the average-peer performance at each school. The treatment dose d is measured by the average transition score among those students that are assigned to that school. In translating this data to a RDD world, the cutoff for admission into a given school is computed by the minimum transition score among the students that are assigned to that school. For most students in the sample, we observe their score on the so called “baccalaureate exam” taken at the end of high school. The score on the baccalaureate exam is our outcome variable Y .

The ranking of preferences for high schools submitted by each student is not observed in the data which makes our RDD fuzzy. For example, a student could have a score greater than the cutoff for the school with highest d in her town but still be assigned to a different school because of her preferences. According to the cutoffs computed, 38% of the students in the sample are assigned to the high school with the highest d among those schools in their towns with admission cutoffs less or equal than their transition scores. If students had exactly the same rank of school preferences in each town, the RDD would be sharp and every student would attend the school with highest d among those with admission cutoffs less or equal than her score. Even in the fuzzy RDD case, the sharp treatment effect parameters of sections 2, 3, and 4.1 have the Intent-to-Treat (ITT) interpretation: they measure the average academic return of having access to a better school but not necessarily attending it. The fuzzy treatment effect parameters of section 4.2 measures the academic return of going to a better school averaged over the group of compliers.

We compute the admission cutoffs and average peer-performance for each high school in each town-year. For a few town-years, the ordering of schools by admission cutoff does not correspond to the ordering by treatment dose d , a consequence of the fact that students’ preference rankings over schools don’t always coincide with the ranking of schools by average peer-performance d . We are interested in the effect of gaining access to a better school, so we only keep the cutoffs of those schools whose d is higher than the schools with smaller cutoffs. Also, we merge a few schools that happen to have the same admission cutoff in some town-years. These procedures lead to a monotonically increasing treatment schedule

for every town-year $p \in \mathcal{P}$ $c_{p,j-1} < c_{p,j}$ and $d_{p,j-1} < d_{p,j}$. Once we drop the observations with missing values for the baccalaureate exam, we are left with a total of 237062 students, 826 schools, 131 towns, and 939 cutoffs. Figure 2(a) illustrates the distribution of the number of cutoffs across town-years. The asymptotic distributions derived in this paper assume independence of first-step estimates across cutoffs. Independence across cutoffs is mimicked in the finite sample by matching each individual observation to one single cutoff. In other words, each cutoff has a maximum estimation window around it such that windows do not overlap across cutoffs. For the majority of cutoffs in the sample, there are enough observations for feasibility of first-step estimation (Figure 2(b)).

Figure 2: Number of Cutoffs and Observations per Cutoff



Notes: (a) Histogram of the number of cutoffs per town-year: in each town-year, the admission cutoff of a given school is the minimum transition score among the students that are assigned to that school. Students' unobserved preference rankings over schools don't always coincide with the ranking of schools by average peer-performance d . This is confirmed by the finding that the ranking of cutoffs does not match the ranking of the respective treatment doses d associated with the school of each cutoff. We only keep those cutoffs that grant access to a high school of higher quality.

(b) Histogram of the number of observations per cutoff: these are the observations used in the first-step estimation of $B_{p,j}$ for each cutoff (p, j) . The neighborhoods around the cutoffs do not overlap and each individual observation in the sample is matched to only one cutoff.

For ease of exposition, we impose a restriction on the treatment effect function

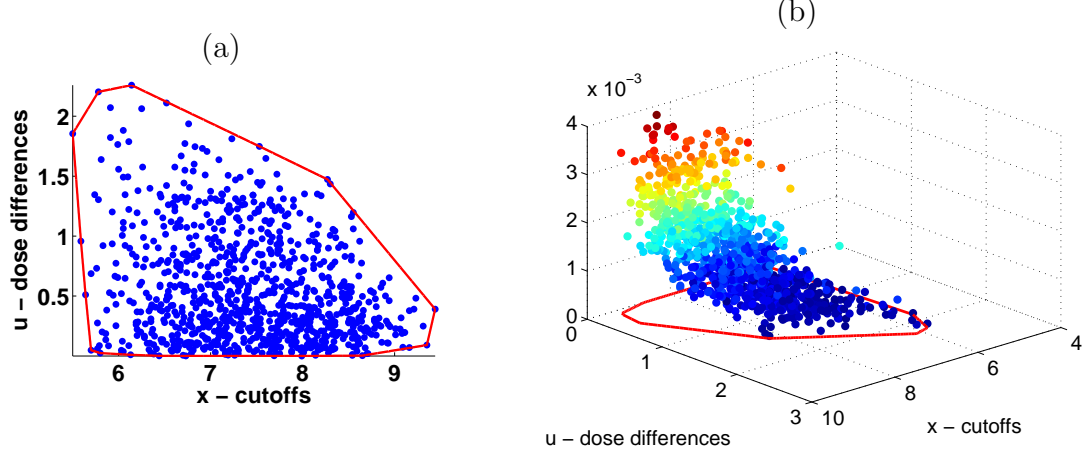
$$\beta(x, d, d') \equiv \beta(x, \underbrace{d' - d}_u) = \beta(x, u)$$

to reduce the dimension of its domain and make possible illustrations of functions of (x, u) in 3D graphs.¹⁴ Figure 3(a) illustrates the variation in cutoff and dose-difference values (x, u) for the Romanian data. The convex hull of $\{(x_{p,j}, u_{p,j})\}_{p \in \mathcal{P}, j \in \mathcal{J}_p}$ is our set \mathcal{C} over which we compute average effects.

The common practice normalizes all cutoffs to zero and use the RDD estimator for one cutoff. Treatment effects are very likely to be heterogeneous, and the normalization procedure estimates a weighted average of local treatment effects weighted by the relative density of individuals near each of the cutoffs (Cattaneo, Keele, Titiunik, and Vazquez-Bare (2014)). Although such implicit weighting scheme is often ignored in applied work, the interpretation of the ALTE depends crucially on how local treatment effects are combined. Figure 3(b) shows the relative density of individuals around each cutoff, that is, $f_{X,P}(c_{p',j'}, p') / \sum_{p,j} f_{X,P}(c_{p,j}, p)$ for every (p', j') in the set \mathcal{C} of figure 3(a). This is the implicit weighting scheme of the ALTE obtained by the normalization procedure.

¹⁴This assumption restricts the returns of going to a better school to be linear in the average peer-performance d . The theory developed in this paper is general enough to deal with the three dimensional domain of β where returns to school quality do not have to be linear in d .

Figure 3: Cutoff-dose Values and Implicit Weighing of Normalization Procedure



Notes: (a) Scatter plot of cutoff and dose-difference values $\{(x_{p,j}, u_{p,j})\}_{p \in \mathcal{P}, j \in \mathcal{J}_p}$ for the 939 cutoffs from the Romanian data. The line that envelops the scatter plot is the convex hull of the set of cutoff-dose values or the set \mathcal{C} over which we compute ATEs.

(b) Three dimensional scatter plot of the implicit weighting scheme of the normalization procedure plotted over set \mathcal{C} . The Z-axis is the relative density of the forcing variable $f_{X,P}(c_{p',j'}, p') / \sum_{p,j} f_{X,P}(c_{p,j}, p)$ for every (p', j') in the set \mathcal{C} .

In table 2, we compare the estimate obtained by the normalization procedure to estimates of ALTE using two different weighting schemes (i) and (ii). Weighing scheme (i) is the relative density of the forcing variable at the existing cutoffs which is the implicit weighting scheme of the normalization procedure. This justifies the similar results of lines 1 and 2 of table 2. The average return of having access to a better high school for those students near the cutoffs is about 0.04 of a point in the bacalaureate exam grade (grades vary between 0 and 10). None of the estimates throughout this section are bias corrected because the bias terms are negligible. Weighing density (ii) is based on a policy question described below.

Suppose that Romania wants to invest in elite high schools as part of a national science and innovation program. The new policy marginally increases the number of vacancies of high schools that currently admit the top 25% students, that is, those students with transition scores greater than 8.69. The goal is to grant high ability students access to better schools by marginally decreasing the admission cutoffs of elite high schools. Opponents to the schools'

expansion argue that top quartile students who would be allowed into the most elite schools would not benefit sufficiently to justify the costs of modifying school buildings, transferring teachers, and the like. We are interested in the average effect among those students that are granted access to better high schools because of this policy. The ALTE of interest weights those individuals that are local to the existing cutoffs of the selected high schools, and its estimate is shown in table 2, third line. The impact of this policy is statistically equal to zero which strongly suggests that local treatment effects are heterogeneous, high ability students won't benefit from having access to better schools, and conclusions based on the normalization procedure are misleading to answer this policy question. The difference in μ_{ALTE} due to different weighting scheme is statistically different than zero (table 2, 4th line).

Table 2: Heterogeneity Case I - Different Weighing Schemes

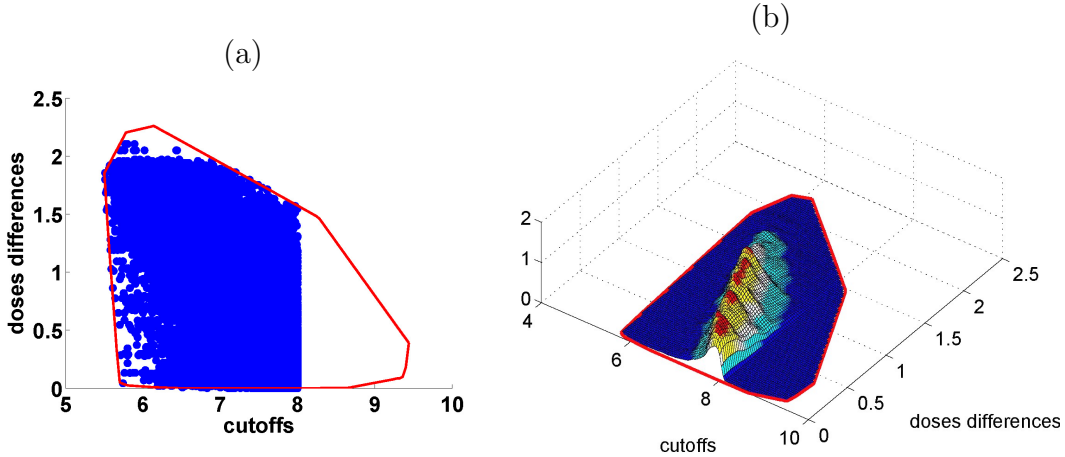
Method	Parameter	Estimate	S.E.
Normalization	$\mu_{ALTE}^{(i)}$	0.0381	0.0082***
Two-step	$\mu_{ALTE}^{(i)}$	0.0417	0.0113***
	$\mu_{ALTE}^{(ii)}$	-0.0243	0.0282
	$\mu_{ALTE}^{(i)} - \mu_{ALTE}^{(ii)}$	0.0659	0.0276**

Notes: ‘Normalization’ pools data from all town-years where each individual is matched to his nearest admission cutoff and the cutoff value is subtracted from the individual transition scores X_i , so to have one cutoff at zero for everybody. Each individual observation is used only once in the estimation (no overlapping estimation windows). Normalization estimates are obtained by Local Linear Regression (LLR) with optimal IK bandwidth (Imbens and Kalyanaraman (2012)) and Edge kernel. Estimates for lines 2,3 and 4 are obtained according to the 2-step estimation procedure described in section 2. The first step uses LLR with IK bandwidth for almost all cutoffs, and the Nadaraya-Watson (i.e. $\rho = 0$) for those few cutoffs that did not have enough observations to run a LLR. The weighting scheme (i) is $\omega_{p',j'} = \hat{f}_{X,P}(c_{p',j'}, p') / \sum_{p,j} \hat{f}_{X,P}(c_{p,j}, p)$ for every p', j' , where $\hat{f}_{X,P}(c_{p',j'}, p')$ is estimated using a uniform kernel with the Silverman's bandwidth. The weighting scheme (ii) is $\omega_{p',j'} = \hat{f}_{X,P}(c_{p',j'}, p') / \sum_{p,j : c_{p,j} \geq 8.96} \hat{f}_{X,P}(c_{p,j}, p)$ for p', j' such that $c_{p',j'} \geq 8.96$ and $\omega_{p',j'} = 0$ otherwise.

The inference method developed under heterogeneity case II allows for estimation of ATEs over continuous counterfactual distributions of the forcing variable and dose-changes. This permits inference over a much more general set of counterfactuals and not only those

policies that target the individuals near the existing cutoffs. Suppose we are interested in using the Romanian data to predict how students would benefit from a new charter school that admits students from disadvantaged backgrounds. More specifically, suppose the charter school admits students by lottery drawing from the national distribution of students with transition score below 8 and that are currently attending a high school of average peer-performance less or equal than 8. Assume that, because the new charter school has more autonomy and better management than traditional public schools, it will be equivalent to a high school of average peer performance equal to 8 (even though its average student scores less than 8). Given these parameters, we compute the distribution of transition scores X , and dose changes $U = 8 - D$ of those individuals admitted into this charter school. Figure 4 illustrates the weighting density $\omega(x, u)$ implied by this policy counterfactual. Note that the support of $\omega(x, u)$ involves not only individuals with transition scores equal to the observed cutoff dose-change values but also away from them (compare figures 3(a) and 4(a)).

Figure 4: Weighing Density of Charter School Example



Notes: (a) The contour line indicates the boundary of the set \mathcal{C} . The shaded region inside set \mathcal{C} is made of the scatter plot of the transition scores and dose-change values of those individuals admitted into the charter school. The shaded area illustrates that the support of the weighting density $\omega(x, u)$ is more general than simply the cutoff dose-change values observed in the data (figure 3(a)).

(b) The weighting density $\omega(x, u)$ implied by the charter school example.

We compute the estimate for μ_{ATE} using the distribution $\omega(x, u)$ of students admitted

into the charter school (table 3). The estimate $\hat{\mu}_{ATE}$ is equal to 0.0547 of a point in the baccalaureate exam grade, and it is statistically different than zero. Estimation of μ_{ATE} requires corrected weights as discussed in section 3. A natural question that arises is how well the parameter μ_{ALTE} , defined using the relative density weights from the charter school density $\omega(x, u)$, approximates the parameter μ_{ATE} . The estimate $\hat{\mu}_{ALTE}$ computed in this manner is approximately half of $\hat{\mu}_{ATE}$ and statistically insignificant. Also, under finite K asymptotics, the null hypothesis of equality between $\mu_{ALTE}^{(ii)}$ defined using relative density weights and $\mu_{ALTE}^{(i)}$ defined using corrected relative density weights is rejected at 5% significance (third line of table 3). The charter school policy question demands an ATE computed over the entire distribution of students admitted. Using the distribution of student near the existing cutoffs in the Romanian data leads to misleading conclusions. The difference between μ_{ATE} and μ_{ALTE} arises from non-linearities in the treatment effect function $\beta(x, u)$ that are not captured in μ_{ALTE} because it averages over only those individuals that are local to existing cutoff values.

Table 3: Heterogeneity Case II - Charter School Example

Parameter	Estimate	S.E.
μ_{ATE}	0.0547	0.0187***
$\mu_{ALTE}^{(ii)}$	0.0283	0.0176
$\mu_{ALTE}^{(i)} - \mu_{ALTE}^{(ii)}$	0.0264	0.0135**

Notes: The first step estimation of both μ_{ALTE} and μ_{ATE} uses LLR with IK bandwidth for almost all cutoffs, and the Nadaraya-Watson ($\rho = 0$) for those few cutoffs that did not have enough observations to run a LLR. The second step estimation of μ_{ALTE} averages first step estimates $\hat{B}_{p,j}$ using the weighting scheme $\omega_{p',j'} = \omega(c_{p',j'}, p') / \sum_{p,j} \omega(c_{p,j}, p)$, where $\omega(x, u)$ refers to the charter school density. The second step estimation of μ_{ATE} uses a bivariate local cubic regression ($\rho_I = 3$) to compute the corrected weights $\Delta_{p,j}$. Corrected weights depend on a choice of second step bandwidth h_I , and h_I was chosen to minimize the MSE of $\hat{\mu}_{ATE}$. Using the charter school density $\omega(x, u)$, we define $\mu_{ALTE}^{(i)}$ with corrected relative density weights, and $\mu_{ALTE}^{(ii)}$ with relative density weights.

We illustrate our estimation procedure for heterogeneity case III by specifying the fol-

lowing parametric functional form for the treatment effect function.

$$\beta(x, u) = \theta_1 u + \theta_2 x u + \theta_3 x^2 u$$

This functional form is chosen to be linear in u in order to be consistent with the restriction that $\beta(x, d, d') = \beta(x, d' - d) = \beta(x, u)$ that was imposed in the beginning of this section. The quadratic term in the transition score x allows for varying marginal effects of ability on the returns to school quality. In table 4, we report the estimates for the θ s and μ_{ATE} for the charter school weighting density. We compare estimates for two choices of weighting matrix Ω : (i) cutoffs are equally weighted; (ii) cutoffs are weighted by the inverse of their first step variance $\mathcal{V}_{p,j}$ (optimal weighting that minimizes variance). The precision of the parameter estimates is greatly improved when the optimal weighting is used, and all parameter estimates become statistically significant. Thetas that are different zero suggest heterogeneity of treatment effects across cutoffs. According to this parametric functional form, the ATE for the charter school example is positive, significant and higher than before.

Table 4: Heterogeneity Case III - Sharp Case

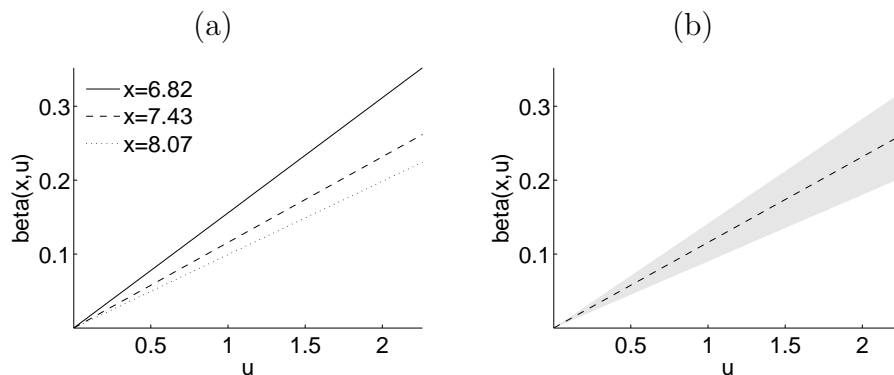
Parameter	Equal Weighting		Optimal Weighting	
	Estimate	S.E.	Estimate	S.E.
θ_1	0.4802	1.2856	2.2085	0.5158***
θ_2	-0.1243	0.3509	-0.5171	0.1465***
θ_3	0.0100	0.0238	0.0317	0.0103***
μ_{ATE} (charter school)	0.0790	0.0133***	0.1147	0.0084***

Notes: the first step estimates of $B_{p,j}$ were obtained by LLR with IK bandwidth for almost all cutoffs, and the Nadaraya-Watson ($\rho = 0$) for those few cutoffs that did not have enough observations to run a LLR. Second step parametric estimates were obtained according to the procedure described in section 4.1 for two choices of weighting matrix Ω : (i) equal weighting, $\Omega = I_{K \times K}$ (identity matrix); (ii) optimal weighting, $\Omega = \text{diag}\{\hat{\mathcal{V}}_{p,j}\}_{p,j}$. The average μ_{ATE} is the integral of the estimated parametric $\beta(x, u)$ weighted by the charter school weighting density $\omega(x, u)$.

Returns to better schooling are increasing in the change in school quality u , and the slope is larger for students with lower transition score. Figure 5(a) plots the treatment effect

function $\beta(x, u)$ against u for the 25th, 50th, and 75th quantiles of the transition scores in set \mathcal{C} . Figure 5(b) repeats the plot of $\beta(x, u)$ for the median value of x and adds 95% pointwise confidence intervals.

Figure 5: Returns to Better Peers and Change in Treatment Dose

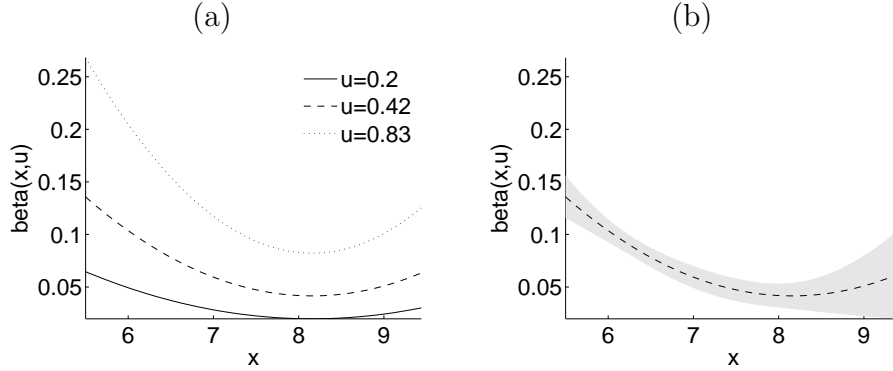


Notes: (a) estimated parametric treatment effect function $\beta(x, u)$ plotted against changes in average peer performance u within set \mathcal{C} . Transition score x is fixed at three choices corresponding to the 25th, 50th and 75th percentiles of the distribution of cutoff values.

(b) For x fixed at the median cutoff value, $\beta(x, u)$ is plotted against u along with 95% confidence bands.

Returns to better schooling generally decrease as the transition score x increases, as shown in figure 6. Panel (a) plots the treatment effect function $\beta(x, u)$ against x for three fixed values of dose changes that are equal to the 25th, 50th and 75th quantiles of the dose-changes in the set \mathcal{C} . Panel (b) plots 95% confidence bands around the $\beta(x, u)$ for the median value of u .

Figure 6: Returns to Better Peers and Transition Score



Notes: (a) estimated parametric treatment effect function $\beta(x, u)$ plotted against transition score x within set \mathcal{C} . The change in average peer performance u is fixed at three choices corresponding to the 25th, 50th and 75th percentiles of the distribution of changes in average peer performance.

(b) For the median value of u , $\beta(x, u)$ is plotted against x along with 95% confidence bands.

Thus far the analysis has been restricted to ITT effects; that is, the average effect on the baccalaureate exam grade when students have access a high school with better peers. Using the parametric functional form specification from above, we turn to the fuzzy case where inference is conditioned on the subgroup of ‘compliers’. We compare estimates for two choices of weighting matrix Ω : (i) cutoffs are equally weighted; (ii) cutoffs are optimally weighted to minimize variance. Similar to the sharp case, the optimal choice of Ω reduces the standard-errors and changes the conclusion of the individual significance tests on the thetas (Table 5). The marginal effects of x and u on treatment effects have a similar shape to the ones in the sharp case (figures 5 and 6), so we don’t plot them again. The ATE over students admitted to the charter school is much higher than the ITT ATE. Attending the better charter school has an impact on the baccalaureate grade of 0.07 point higher than the impact from only having access to the better charter school.

Table 5: Heterogeneity Case III - Fuzzy Case

Parameter	Equal Weighting		Optimal Weighting	
	Estimate	S.E.	Estimate	S.E.
θ_1^c	-0.2458	1.6576	7.3906	0.7253***
θ_2^c	0.1202	0.4362	-1.7812	0.1953***
θ_3^c	-0.0093	0.0285	0.1080	0.0130***
μ_{ATE}^c (charter school)	0.1034	0.0176***	0.1868	0.0106***

Notes: the first step estimates of $B_{p,j}$ were obtained by LLR with IK bandwidth for almost all cutoffs, and the Nadaraya-Watson ($\rho = 0$) for those few cutoffs that did not have enough observations to run a LLR. Second step parametric estimates were obtained according to the procedure described in section 4.2 for two choices of weighting matrix Ω : (i) equal weighting, $\Omega = I_{K \times K}$ (identity matrix); (ii) optimal weighting, $\Omega = \text{diag}\{\hat{V}_{p,j}\}_{p,j}$. The average μ_{ATE}^c is the integral of the estimated parametric $\beta_c(x, u)$ weighted by the charter school weighting density $\omega(x, u)$.

6 Conclusion

Regression discontinuity designs (RDD) have been used in a wide range of applications in Economics since the late 1990s. Identification and estimation results are well developed for the one cutoff case. More recently we see an increasing number of applications with one forcing variable and multiple cutoffs assigning individuals to heterogeneous treatments. There is a lack of theoretical studies investigating the conditions under which researchers can combine multiple local treatment effects to estimate an average treatment effect (ATE). A common practice is to normalize all cutoffs to zero and use the one cutoff estimator to obtain a summary effect. The average effect of the normalization strategy can lead to misleading conclusions if interest lies on average effects with different distributions of individuals including individuals away from existing cutoffs.

This paper proposes inference procedures for average effects in RDD with multiple thresholds. Our estimator is consistent and asymptotically normal for an average treatment effect over the entire support in which we observe variation in cutoffs and treatment doses. If treatment effects follow a non-parametric model, asymptotic results require both the number of observations and cutoffs to grow large. The rate of growth of the number of cutoffs relative to the number of observations determines the feasible set of bandwidth choices. The number of cutoffs cannot grow too fast to allow consistent estimation of local treatment effects uniformly across cutoffs. The number of cutoffs cannot grow too slowly to control the bias in the integral approximation of the ATE. The maximum rate of convergence of the estimator is root-n within the feasible set of bandwidth choices. If treatment effects follow a parametric model, then observations can be optimally combined for efficiency, and a parametric function form obtains identification in the fuzzy case.

We apply our methods to the data of PEU on high school assignment in Romania based on transition scores of students. We examine estimates for two types of average effects: (i) average of local treatment effects (ALTE); and (ii) average treatment effect (ATE). For the (ALTE), we compare the weighting scheme implicit to the normalization strategy (relative

density of individuals at the cutoff values) to a weighting scheme over the top 25% of individuals in the distribution of transition scores. Statistically different average effects illustrate the heterogeneity of local treatment effects and the inability of the normalization strategy to answer policy questions that lead to different weighting schemes. We estimate the ATE over a distribution of individuals admitted into a fictitious charter school. Results indicate the inability that an (ALTE) has to predict the effect of such policy.

We also illustrate estimates of a simple parametric specification that allows returns to better schooling to vary with the transition score and school quality. We find that the optimal weighting scheme that minimizes variance changes inference conclusions on individual parameters. Returns to better schooling are increasing in school quality but at a decreasing rate in transition score. The high school assignment in Romania translates into a fuzzy RDD, so we use the parametric specification to infer the effects on compliers. We find that the average return of going to the charter school for compliers is almost twice as big as the average return of having access to the charter school.

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www.stanford.edu/~bertanha/Bertanha_JMP_appendix.pdf