Does Incomplete Spanning in International Financial Markets Help to Explain Exchange Rates?

Hanno Lustig
Stanford GSB and NBER

Adrien Verdelhan
MIT Sloan and NBER

January 2018*

Abstract

We assume that domestic (foreign) agents, when investing abroad, can only trade in the foreign (domestic) risk-free rates. In a preference-free environment, we derive the exchange rate volatility and risk premia in any such incomplete spanning model, as well as a measure of exchange rate cyclicality. We find that incomplete spanning lowers the volatility of exchange rate, increases the risk premia but only by creating exchange rate predictability, and does not affect the exchange rate cyclicality.

Keywords: Exchange rates, Incomplete Markets, Currency Risk Premia.

*First Version: April 2015. Lustig: Stanford Graduate School of Business, 355 Knight Way, Stanford, Ca 94305 (hlustig@stanford.edu). Verdelhan: MIT Sloan School of Management, 100 Main Street, E62-621, Cambridge, MA 02139 (adrienv@mit.edu). We are grateful to Mark Aguiar, Emmanuel Farhi, Jack Favilukis, Bob Hodrick, Robert Richmond, and Alexandros Vardoulakis for their discussions and detailed comments, and we would like to thank Bob Hodrick for conversations that led to this paper. We are grateful to John Cochrane, Ric Colacito, Max Croce, Bernard Dumas, Ana Fostel, Xavier Gabaix, Francisco Gomes, Ben Hebert, Leonid Kogan, Sylvain Leduc, Matteo Maggiori, Jonathan Parker, Anna Pavlova, Carolin Pflueger, Anders Trolle, Raman Uppal and seminar participants at different institutions and conferences for their comments.

1 Introduction

Our paper focuses on three key exchange rate puzzles: the volatility puzzle of Brandt, Cochrane, and Santa-Clara (2006), the cyclicality puzzle of Backus and Smith (1993), and the currency risk premium puzzle of Fama (1984). Along these three dimensions, the data are at odds with the standard international business cycle and international asset pricing models that typically assume that financial markets are complete. In recent years, keeping the assumption that markets are complete, a series of papers propose compelling resolutions of these three puzzles, based on the long-run risk, the habit, or the disaster risk frameworks. While the models differ, they all assume that SDFs are highly correlated across countries — a feature that is difficult to test directly in the data because SDFs in these models depend on hard-to-measure long-run consumption, habit ratios, or disaster probabilities. In this paper, we pursue another route, relaxing the complete spanning assumption.

To what extent can incomplete spanning resolve the core exchange rate puzzles? We consider an extreme case, where the domestic (foreign) agents, when investing abroad, can only trade in the foreign (domestic) risk-free rates. We find that even this extreme departure from complete spanning has limits: it can help match quantitatively the volatility of exchange rates in the data and the currency risk premium, but it has no impact on a Backus-Smith measure of exchange rate cyclicality.

To derive this preference-free result, we make two key assumptions: 1) the existence of a log-normal stochastic discount factor in the space of traded assets in each country, 2) the existence of a domestic and a foreign risk-free rate in which the other country's investors can invest. With these two assumptions, we can derive closed-form expressions for the exchange rate volatility, cyclicality, and risk premia that are valid in any incomplete spanning model.

To do so, for any pair of log SDFs m and m^* that price the domestic and foreign assets from the perspective of respectively the domestic and foreign investors, we introduce a wedge η that reconciles the log change in exchange rates with the domestic and foreign SDFs: $\Delta s = \eta + m^* - m$. These stochastic wedges, first introduced by Backus, Foresi, and Telmer's (2001), can be interpreted as ratios of tax rates on exchange rate transactions that mimic the effects of market incompleteness in a complete markets world. Similar approaches have been used in the macroeconomics literature, most notably in Chari, Kehoe, and McGrattan (2007). Complete markets is a special case, where the wedge is zero.

In our framework, four Euler equations have to hold simultaneously: the domestic investor can buy the domestic risk-free asset and the foreign risk-free asset (bearing there the exchange rate risk), while the foreign investor can buy the foreign risk-free asset, as well as the domestic one (here again bearing the exchange rate risk). This is equivalent to assuming that foreign investors have unconstrained access to one-period ahead forward currency markets, but perhaps not to other asset markets. If domestic investors had access to more foreign assets, additional Euler equations would apply. Our four Euler equations imply some restrictions on the moments of the wedges η, and these restrictions have some bite.

We find that the Euler equation restrictions imply that the volatility of exchange rates always decreases relative to the complete markets benchmark. In a lognormal world, the higher the volatility of the wedge, the lower the volatility of the exchange rate changes, thus helping to resolve the volatility puzzle. After matching the exchange rate volatility, the currency risk premium depends only on the first moment of the wedge. To increase the currency risk premium, incomplete spanning models need to introduce a predictable component in the exchange rate changes. The impact of the Euler equation restrictions on the exchange rate cyclicality is even starker.

To maintain our preference-free approach, we define the exchange rate cyclicality by the comovement of exchange rate changes with the difference in SDFs, $m^* - m$. The interpretation in terms of cyclicality follows when stochastic discount factors are large in bad times, as for example in most existing macro-finance models. In a representative agent model with power utility, the cyclicality would thus be measured by the comovement between exchange rate changes and relative consumption growth rates, as in the seminal Backus and Smith (1993) contribution. We find that, in a regression of the difference in log SDFs on the log changes in exchange rates, the slope coefficient is one, as in complete markets. In other words, the home currency depreciate when the home investor experience better times than the foreign investor. Incomplete spanning models that satisfy our two assumptions cannot address the cyclicality puzzle.

To confront our theoretical results with the data, we make one additional assumption: the log domestic and foreign SDFs, m and m^*, are volatile, in order to reproduce the Sharpe ratio on aggregate equity and currency markets. We consider all possible cross-country correlations between m and m^*. As an example, the volatility of the log SDF has to be around 50% to match the equity risk premium, and a cross-country correlation of 0.5 would be a large value considering the cross-country correlation of any macroeconomic variables.

With these volatile SDFs, we turn to the exchange rate volatility, risk premium, and cyclicality. First, the volatility of the wedge has to be very close to the volatility of the log SDFs in order to match the exchange rate volatility. When the cross-country correlation of the SDFs is 0.99, there is no need to introduce any wedge to match the volatility of exchange rate. But for any cross-country correlation of SDFs below 0.8, the wedge has to exhibit a volatility of at least 30%. Second, in a model where the domestic and foreign log SDFs have the same volatility, the log currency risk premium is zero when markets are complete. Introducing a drift of 4% in the wedge thus help match the currency risk premium. This result is valid for any correlation of the log SDFs: once the model matches the exchange rate volatility, the currency risk premia in logs as in levels only depend on the first moment of the wedge. To obtain a large currency risk premium without exchange rate predictability, one needs to start with some home and foreign SDFs of different volatilities. Third, the Backus-Smith slope coefficient is one, as already noted, no matter the SDF volatilities or SDF correlations or wedges. By increasing the currency risk premium, the wedge then naturally increases, in absolute values, the covariance between the SDF and the exchange rate changes. In this sense, cyclicality even worsens: exchange rates tend to decrease in bad times, even more so than in complete markets.

Our results so far pertain to a log-normal world. They are not, however, a simple rejection of log-normality. Relaxing our first assumption and considering higher moments of the wedges and SDFs, we derive restrictions on currency wedges in terms of their entropy and co-entropy (as defined in Backus, Chernov, and Boyarchenko, 2016) with the pricing kernels, as well as closed-form expressions for the exchange rate entropy and risk premia. Even in this general case, increases in the volatility of the exchange rate changes go hand in hand with decreases in currency risk premia. The only way to counteract the decrease of the risk premium brought about by the decrease in exchange rate volatility is to impute a large non-stationary component in the exchange rate changes through a large drift in the stochastic currency wedge.

We end the paper with model-specific examples of our general results, and a look at the incomplete market models through the lenses of our assumptions. We consider the extension of the simple consumption-CAPM model to jumps and to dynamic asset pricing models. While one cannot rule out the existence of a non-Gaussian model that would match the three exchange rate puzzles simultaneously thanks to incomplete spanning, we do not know of such a model. In a calibrated version of the benchmark Merton (1976) and consumption disaster model, however, we find that the introduction of incomplete spanning of the consumption disasters cannot address the three puzzles simultaneously. In a large class of dynamic asset pricing models, the higher-order moments of the currency wedge are related to its first moment by no arbitrage restrictions, always lowering currency market risk premia and Sharpe ratios. Thus, in a large class of models, under our trading assumption, the first moment of the wedge is not a free parameter. Finally, we show that the existing incomplete market models do not satisfy our theoretical or empirical assumptions.

Literature Review The exchange rate volatility, cyclicality, and forward premium puzzles highlight the limits of most international economics models that assume that the menu of contingent claims spans all states of the world, following the seminal work of Lucas (1982). The three puzzles are the subject of a very large literature, with literally hundreds of contributions published in the last thirty years. We briefly review the puzzles before turning to the existing solutions. (i) Hansen and Jagannathan (1991) show that stochastic discount factors have to be highly volatile in order to reproduce observed equity premia. As Brandt, Cochrane, and Santa-Clara (2006) point out, stochastic discount factors must be almost perfectly correlated in order to match the comparatively low exchange rate volatility in the data. But macroeconomic variables exhibit low correlations across countries. (ii) When markets are complete and agents have constant relative risk aversion preferences, changes in exchange rates must be perfectly correlated with relative consumption growth rates in the domestic and foreign economies. As was first pointed out by Kollmann (1991) and Backus and Smith (1993), the low correlation in the data is therefore surprising. (iii) As documented by Tryon (1979), Hansen and Hodrick (1980) and Fama (1984), interest rate differences do not predict subsequent changes in exchange rates, thus giving rise to large deviations from the uncovered interest rate parity condition and currency carry trade risk premia. The size of currency risk premia represents a challenge for many models in international economics.

Colacito and Croce (2011, 2013), as well as Colacito et al. (2014, 2017), Bansal and Shaliastovich (2012), Farhi and Gabaix (2015), Gabaix and Maggiori (2015), and Stathopoulos (2017) address the aforementioned puzzles in models respectively based on long-run risk preferences, rare disaster risk, segmented markets, or habit preferences. The long-run risk models assume or infer that the slow moving components of consumption growth are perfectly correlated across countries.[1] The disaster risk model assumes that exchange rates exhibit a low probability of a large depreciation. The segmented market models assume a very large correlation between exchange rate changes and the consumption growth of the market participants. Since we find that incomplete spanning can only go so far in explaining exchange rate puzzles, our paper implicitly argue in favor of more empirical work along the lines suggested by the recent contributions to the international finance literature.

Building on Brandt, Cochrane, and Santa-Clara (2006) and Backus, Foresi, and Telmer's (2001), our paper also complements a growing literature in international economics and finance that study exchange rates in incomplete markets. There is a wealth of empirical evidence that investors act as if they face an incomplete menu of assets abroad, either because of explicit transactions and capital controls, or because of other frictions (Lewis, 1995). Notable recent theoretical contributions include the work by Alvarez, Atkeson, and Kehoe (2002), Chari, Kehoe,

[1] In these models, the SDFs are volatile enough to reproduce the equity premium and the forward premium puzzles, but, thanks to their long-run risk components, the SDFs are almost perfectly correlated such that exchange rates are as volatile as in the data. Since the volatility of the SDFs is mostly due to the long-run risk components, not to the consumption growth shocks, the correlation between exchange rates and relative consumption growth rates is low, as in the data. The long-run risk components are, however, difficult to measure in the data and most evidence is drawn indirectly from asset prices and not macroeconomic quantities, or are the predictions of models with endogenous diffusion of technology, as in Croce, Nguyen and Schmid (2012) or Gavazzoni and Santacreu (2015).

and McGrattan (2002), Bacchetta and van Wincoop (2006), Corsetti, Dedola, and Leduc (2008), Alvarez, Atkeson, and Kehoe (2009), Pavlova and Rigobon (2010, 2012), Bruno and Shin (2014), Maggiori (2017), Gabaix and Maggiori (2015), and Favilukis, Garlappi, and Neamati (2015). Instead of specifying a fully-fledged international economics model as these authors do, we seek results that are valid for any stochastic discount factors.

The rest of this paper is organized as follows. Section 2 presents our main theoretical results. Section 3 then studies the quantitative implications of our results and thus the ability of incomplete spanning models to match the exchange rate volatility, the currency risk premium, and the exchange rate cyclicality. Section 4 applies our preference-free results to some model-specific examples and compares our assumptions to recent incomplete market models. Section 5 concludes. All the proofs are presented in the Online Appendix.

2 Theoretical Results

In this section, we first define our notation, then derive key restrictions on incomplete market models imposed by the tradability of risk-free bonds, and finally study their implications for three major moments of exchange rates.

2.1 Notation

We start by defining some notation. All variables are functions of states, not just of time, but we use a time subscript as shorthand when clear. We define M and M^* as functions that map the states of nature into the positive real line. Our goal is to place restrictions on the moments of these variables, moments with clear empirical counterparts. To do so, we assume that M and M^* are the domestic and foreign SDFs that satisfy the Euler equations for the domestic and foreign returns:

$$E_t \left(M_{t+1} R_{t+1} \right) = 1, \tag{1}$$

$$E_t \left(M^*_{t+1} R^*_{t+1} \right) = 1, \tag{2}$$

where R^*_{t+1} represents the foreign return expressed in units of foreign currency, while R_{t+1} denotes the domestic return, expressed in units of the domestic currency.[2] More generally, x^* denotes a foreign variable expressed in units of foreign currency.

We then introduce a wedge η that reconciles the log change in exchange rates with the difference in log SDFs. Throughout the paper, lower case letters denote natural logarithms. The log changes of the exchange rate is thus

$$\Delta s_{t+1} = \eta_{t+1} + m^*_{t+1} - m_{t+1}, \tag{3}$$

where S_t denotes the nominal exchange rate in domestic currency (e.g., U.S. dollars) per unit of foreign currency. When S_t increases, the foreign currency appreciates and the U.S. dollar depreciates. We define variables in nominal terms, but a similar analysis applies to real variables.[3] The wedge η_{t+1} links the actual exchange rates to the difference in the log SDFs.

When asset markets are complete, the stochastic discount factor is unique.[4] In this case, the wedge η_{t+1} is 0. To see this clearly, note that when markets are complete, the domestic and

[2]If the law of one price holds in financial markets and investors can form portfolios freely, then a unique stochastic discount factor exists in the space of traded assets (see Ross, 1978; Cochrane, 2005, for a textbook exposition). But there are cases when investors cannot form portfolios freely (e.g., in the presence of short-selling constraints) or when the law of one price in financial markets fails, and thus the existence of a stochastic discount factor is not guaranteed.

[3]Among developed countries, in the absence of high and volatile inflation rates, the three exchange rate puzzles that we study in the paper exist on both nominal and real variables: the volatilities of real and nominal exchange rates are similar, and so are their correlations with macroeconomic variables and their risk premia. When dealing with real variables, in a world with multiple goods, we would choose one good in each country to be the numéraire; S_t would then denote the real exchange rate, expressed in units of the domestic numéraire, per unit of the foreign numéraire; and M_t would be expressed in the domestic numéraire. Our analysis allows for different consumption baskets at home and abroad.

[4]Markets are complete when investors can invest in any contingent claim, either directly or by synthesizing contingent claims using other securities. In other words, markets are complete when securities' payoffs span all the possible states of nature. Suppose that there are N possible states of nature tomorrow. Each contingent claim is a security that pays one dollar in one state n only tomorrow; $pc(n)$ is its price today. Each asset is defined by the set X of its payoffs in each state of nature. Let $x(n)$ denote an asset's payoff in state n, then the asset price $P(X)$ must satisfy:

$$P(X) = \sum_n pc(n)x(n) = \sum_n \pi(n)\frac{pc(n)}{\pi(n)}x(n) = E\left(MX\right), \tag{4}$$

where the last equality is simply a definition of the stochastic discount factor M. When markets are complete, the stochastic discount factor is clearly unique.

foreign investor's Euler equations apply for any foreign return R_t^*:

$$E_t\left(M_{t+1}\frac{S_{t+1}}{S_t}R_{t+1}^*\right) = 1, \tag{5}$$

$$E_t\left(M_{t+1}^* R_{t+1}^*\right) = 1. \tag{6}$$

Since the stochastic discount factor is unique when markets are complete, then $M_{t+1}^* = M_{t+1}\frac{S_{t+1}}{S_t}$, and in logs, $\Delta s_{t+1} = m_{t+1}^* - m_{t+1}$. In models that feature complete spanning, one can thus back out the implied changes in exchange rates from the stochastic discount factors at home and abroad. Equivalently, one can start from the domestic (or foreign) stochastic discount factor and the rate of change in exchange rates, and then derive the implicit foreign (domestic) SDF. When goods markets are not frictionless or domestic and foreign agents consume different goods, then real exchange rates vary in equilibrium even if financial markets are themselves frictionless.

In the case of the Breeden-Lucas-Rubenstein representative agent model with power utility, then the real log pricing kernel is $m_{t+1} = \log\delta - \gamma\Delta c_{t+1}$ where γ denotes the coefficient of relative risk aversion, δ denotes the rate of time preference, and Δc_{t+1} denotes log aggregate consumption growth. Similarly, the foreign log pricing kernel is $m_{t+1}^* = \log\delta^* - \gamma^*\Delta c_{t+1}^*$. The econometrician can test this model by gathering data on aggregate consumption growth at home and abroad. This is, however, only an example: in this paper, we seek to derive model-free results. M and M^* are equilibrium outcomes that vary with the asset structure and the models, but our results are valid for any pair of M and M^*.

Our wedge-based approach comes from the work of Backus, Foresi, and Telmer (2001) who, in their study of currency risk premia, define new 'perturbed' SDFs denoted $\widehat{M}_{t+1}^*$:

$$\widehat{M}_{t+1}^* = M_{t+1}^* \exp(\eta_{t+1}) = M_{t+1}\frac{S_{t+1}}{S_t}. \tag{7}$$

When markets are incomplete, the stochastic discount factor is not unique: under certain conditions, a unique stochastic discount factor exists in the space of traded assets, but many others potentially exist outside that space. Each wedge η_{t+1} thus defines a potential stochastic discount factor. This alternative view is equivalent to our approach, where we start from any pair of SDFs and define the wedge to match the exchange rate. In Section E of the appendix, we show how our results do not change when we project the SDFs onto the space of traded payoffs, which includes the domestic and the foreign risk-free, and then use projections in the analysis. The exchange rate depends on the wedge, and hence fundamentally determines the space of traded payoffs.

2.2 Assumptions

To show how spanning restrictions in general may help understand exchange rates, we make two assumptions.

Assumption 1. *We assume that the log domestic and foreign stochastic discount factors, m and m^*, and the wedge η are jointly normal.*

The lognormality assumption delivers clear closed-form solutions. We relax this assumption at the end of this section to derive more general results.

Assumption 2. *We assume that there exists a risk-free asset at home and abroad that can be bought and sold by both domestic and foreign investors.*

Risk free bonds are freely traded. This is our key assumption. Domestic investors can trade the foreign risk-free asset, and vice-versa, the foreign investor can invest in the domestic risk-free asset. In other words, Assumption 2 is reminiscent of the covered interest parity: if investors can invest at the domestic and foreign risk-free rates, then forward rates (scaled by the current spot rate) are simply equal to the interest rate differences across countries. The covered interest parity condition is a very accurate description of the data, up to but not after the recent financial crisis (see Du et al., 2017). All the puzzles we study, however, existed before the recent financial crisis, at a time when the covered interest parity held tightly in the data. This is not to say that Assumption 2 is necessarily verified in practice or in all models: risk-free rates may not exist, notably in emerging markets, and capital controls or other frictions may prevent the covered interest parity arbitrage, as notably in the recent work of Gabaix and Maggiori (2015), Schmitt-Grohé and Uribe (2016), Farhi and Werning (2017), Amador et al. (2017), and Itskhoki and Mukhin (2017). These incomplete market models do not satisfy Assumption 2.

We turn now to the implications for the wedge η of the Assumptions 1 and 2.

2.3 The Restrictions on the Wedge

Based on our definition of exchange rates, the domestic investor's Euler equation for foreign assets, and the foreign investor's Euler equation for the domestic assets are

$$E_t\left(M_{t+1}\frac{S_{t+1}}{S_t}R^*_{t+1}\right) = E_t\left(M^*_{t+1}\exp(\eta_{t+1})R^*_{t+1}\right) = 1, \tag{8}$$

$$E_t\left(M^*_{t+1}\frac{S_t}{S_{t+1}}R_{t+1}\right) = E_t\left(M_{t+1}\exp(-\eta_{t+1})R_{t+1}\right) = 1. \tag{9}$$

Since the risk-free payoffs are in the space of traded assets for all investors, domestic and foreign, the risk-free returns satisfy not only the Euler Equations (1) and (2), but also Equations (8) and (9). These four Euler equations applied to risk-free rates impose some conditions on the wedge η, summarized in the following proposition.

Proposition 1. *Under Assumptions 1 and 2, when the exchange rate change is* $\Delta s_{t+1} = \eta_{t+1} + m^*_{t+1} - m_{t+1}$, *then the wedge* η_{t+1} *satisfies:*

$$covar_t\left(m^*_{t+1}, \eta_{t+1}\right) = -E_t\left(\eta_{t+1}\right) - \frac{1}{2}var_t\left(\eta_{t+1}\right), \tag{10}$$

$$covar_t\left(m_{t+1}, \eta_{t+1}\right) = -E_t\left(\eta_{t+1}\right) + \frac{1}{2}var_t\left(\eta_{t+1}\right), \tag{11}$$

where $E_t\left(\eta_{t+1}\right)$ *satisfies these additional restrictions:*

$$\begin{aligned}
-E_t\left(\eta_{t+1}\right) &\leq std_t\left(\eta_{t+1}\right)\left(std_t\left(m^*_{t+1}\right) + \frac{1}{2}std_t\left(\eta_{t+1}\right)\right), \textit{when } E_t\left(\eta_{t+1}\right) \leq -\frac{1}{2}var_t\left(\eta_{t+1}\right),\\
E_t\left(\eta_{t+1}\right) &\leq std_t\left(\eta_{t+1}\right)\left(std_t\left(m^*_{t+1}\right) - \frac{1}{2}std_t\left(\eta_{t+1}\right)\right), \textit{when } E_t\left(\eta_{t+1}\right) \geq -\frac{1}{2}var_t\left(\eta_{t+1}\right),\\
E_t\left(\eta_{t+1}\right) &\leq std_t\left(\eta_{t+1}\right)\left(std_t\left(m_{t+1}\right) + \frac{1}{2}std_t\left(\eta_{t+1}\right)\right), \textit{when } E_t\left(\eta_{t+1}\right) \geq \frac{1}{2}var_t\left(\eta_{t+1}\right),\\
-E_t\left(\eta_{t+1}\right) &\leq std_t\left(\eta_{t+1}\right)\left(std_t\left(m_{t+1}\right) - \frac{1}{2}std_t\left(\eta_{t+1}\right)\right), \textit{when } E_t\left(\eta_{t+1}\right) \leq \frac{1}{2}var_t\left(\eta_{t+1}\right),\\
std_t\left(\eta_{t+1}\right) &\leq std_t\left(m^*_{t+1} - m_{t+1}\right), \textit{everywhere.}
\end{aligned}$$

Hence, there are limits as to how much incomplete markets noise we can introduce. For example, when the exchange rate wedge has a conditional mean of zero ($E_t(\eta_{t+1})$), the amount of noise is bounded above by the following two conditions:

$$2min(std_t(m_{t+1}), std_t(m^*_{t+1})) \geq std_t(\eta_{t+1}) \text{ and } std_t\left(m^*_{t+1} - m_{t+1}\right) \geq std_t(\eta_{t+1}).$$

The upper bound on the wedge's volatility has a simple economic counterpart. Recall that the standard deviation of the log SDF measures the maximal Sharpe ratio among traded assets, defined as the ratio of expected excess returns to their volatilities. This well-known result derives directly from the Euler equation in a lognormal world; $E_t(M_{t+1}R_{t+1}) = 1$ implies that:

$$\frac{E_t\left(R_{t+1} - R^f_t\right)}{std_t(R_{t+1})} \leq std_t(m_{t+1}), \text{for any return } R_{t+1}. \tag{12}$$

The maximal volatility of the wedge is thus linked to the maximal Sharpe ratio among traded assets.

Naturally, if foreign investors can invest in other domestic assets, this will give rise to additional restrictions on the wedges. For example, if the foreign investor's can trade an additional risky asset, then the wedges cannot covary with these risky returns (r_{t+1}, r^*_{t+1}):

$$covar_t\left(r^*_{t+1}, \eta_{t+1}\right) = 0 = covar_t\left(r_{t+1}, \eta_{t+1}\right). \tag{13}$$

Likewise, in dynamic asset pricing models, the drift term of the wedge imputed to the exchange rate process is not a free parameter but is instead determined by no arbitrage condition. We ignore these additional constraints, which may further limit the explanatory power of incomplete spanning models. Instead, we now turn to three key moments of exchange rates, volatility, risk premia, and cyclicality, and in each case derive their values in incomplete market models that let agents trade risk-free bonds.

2.4 Key Theoretical Implications

First, we find that the volatility of the exchange rate decreases relative to the complete spanning benchmark one-for-one with the volatility of the wedge, as noted in the following corollary.

Corollary 1. *Under Assumptions 1 and 2, when the exchange rate change is* $\Delta s_{t+1} = \eta_{t+1} + m^*_{t+1} - m_{t+1}$*, the volatility of exchange rates is given by:*

$$var_t(\Delta s_{t+1}) = var_t(m_{t+1}) + var_t(m^*_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1}) - var_t(\eta_{t+1}).$$

This result follows directly from the covariance restrictions in Equations (10) and (11) that imply $covar_t\left(m^*_{t+1} - m_{t+1}, \eta_{t+1}\right) < 0$. Since the wedges comove negatively with the log difference in SDFs, they offset the effect of m and m^*, and thus reduce the overall volatility of the exchange rate.

Second, we turn to the currency risk premium. When markets are complete, the log currency risk premium established in Bekaert (1996), Bansal (1997), Backus, Foresi, and Telmer (2001) is simply $1/2\left[var_t\left(m_{t+1}\right) - var_t\left(m^*_{t+1}\right)\right]$. When markets are incomplete, the drift of the wedge affects the log currency risk premium, and the volatility of the wedge affects the currency risk premia in levels, as noted in the following corollary.

Corollary 2. *Under Assumptions 1 and 2, when the exchange rate change is* $\Delta s_{t+1} = \eta_{t+1} + m^*_{t+1} - m_{t+1}$*, then the currency risk premium in logs on a long position in foreign currency is:*

$$E_t[rx^{FX}_{t+1}] \equiv r^{f,*}_t - r^f_t + E_t(\Delta s_{t+1}) = \frac{1}{2}\left[var_t\left(m_{t+1}\right) - var_t\left(m^*_{t+1}\right)\right] + E_t\left(\eta_{t+1}\right).$$

The currency risk premium in levels on a long position in foreign currency is given by:

$$\begin{aligned} E_t[rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] &= -cov_t(m_{t+1}, \Delta s_{t+1}) \\ &= var_t\left(m_{t+1}\right) - covar_t\left(m^*_{t+1}, m_{t+1}\right) - \frac{1}{2}var_t\left(\eta_{t+1}\right) + E_t\left(\eta_{t+1}\right). \end{aligned}$$

The currency risk premium in levels, from the perspective of the foreign investor, is given by:

$$\begin{aligned} E_t[-rx_{t+1}^{FX}] + \frac{1}{2}var_t[rx_{t+1}^{FX}] &= -cov_t(m_{t+1}^*, -\Delta s_{t+1}) \\ &= var_t\left(m_{t+1}^*\right) - covar_t\left(m_{t+1}^*, m_{t+1}\right) - \frac{1}{2}var_t\left(\eta_{t+1}\right) - E_t\left(\eta_{t+1}\right). \end{aligned}$$

Market incompleteness wedges can increase currency risk premia, but only at the cost of introducing exchange rate predictability through an increase in the drift.

Third, we turn now to the exchange rate cyclicality. Its usual definition is the correlation of exchange rate changes with consumption, output or employment growth rates. With CRRA preferences, for example, complete markets imply a perfect correlation between relative consumption growth rates and exchange rate changes. To preserve the model-free aspect of our work, we define the exchange rate cyclicality in more general terms, that naturally encompass the usual definition. To do so, we assume that stochastic discount factors are large in bad times, as in all existing macro-finance models. For our purpose, the exchange rate cyclicality is thus measured by the correlation of exchange rates with m^*, m, or their difference. Complete markets imply, perhaps counterintuitively, that the home currency depreciates in relatively good times for home investors. Under Assumptions 1 and 2, incomplete market models are no different.

Corollary 3. *Under Assumptions 1 and 2, when the exchange rate change is* $\Delta s_{t+1} = \eta_{t+1} + m_{t+1}^* - m_{t+1}$*, then the covariance between the difference in log stochastic discount factors* $m^* - m$ *and the change in exchange rates in incomplete markets is non-negative and equal to the variance of exchange rates:*

$$\begin{aligned} covar_t\left(m_{t+1}^* - m_{t+1}, \Delta s_{t+1}\right) &= covar_t\left(m_{t+1}^* - m_{t+1}, \eta_{t+1}\right) + var_t\left(m_{t+1}^* - m_{t+1}\right), \\ &= var_t\left(m_{t+1}^* - m_{t+1}\right) - var_t\left(\eta_{t+1}\right) = var_t\left(\Delta s_{t+1}\right) \geq 0 \end{aligned}$$

As a result, the slope coefficient in a regression of $m^* - m$ *on exchange rate changes is equal to*

1, its value when markets are complete:

$$\beta_{Backus-Smith} \equiv \frac{covar_t\left(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}\right)}{var_t\left(\Delta s_{t+1}\right)} = 1.$$

Only the correlation would decrease:

$$corr_t\left(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}\right) = \frac{\sqrt{var_t(m_{t+1}) + var_t(m^*_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1}) - var_t(\eta_{t+1})}}{\sqrt{var_t(m_{t+1}) + var_t(m^*_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1})}} \leq 1.$$

Complete markets imply a perfect correlation between the difference in log stochastic discount factors and the log change in exchange rates. Corollary 3 shows that the impact of incomplete spanning on measures of exchange rate cyclicality is limited for three reasons. First, incomplete spanning does not change the sign of the covariance between exchange rate changes and the difference in log stochastic discount factors spanned by asset markets. Even in incomplete markets, as soon as agents can trade in risk-free bonds, exchange rates will depreciate when the home investor experience better times than the foreign investor. In a utility-based framework, those times, however, are defined by the marginal utility spanned by asset markets, whereas the total marginal utility of the investor may be high or low. Second, incomplete spanning does not change the slope coefficient in a regression of the difference in log stochastic discount factors on exchange rate changes; it is equal to one, as in complete markets. Third, incomplete spanning decreases only one measure of the cyclicality: the correlation between exchange rates and the stochastic discount factor, and it clearly does it only because of the volatility of the wedge and thus at the cost of a lower Sharpe ratio on the currency risk premium.

2.5 Non-normality

Corolarries 1, 2, and 3 are our key preference-free results. Some of these results can be extended to an environment with non-Gaussian shocks. To do so, we use a different, entropy-based measure of risk. The conditional entropy of a random variable X_{t+1} is equal to: $L_t(X_{t+1}) = \log E_t(X_{t+1}) - E_t(\log X_{t+1})$. If the random variable X_{t+1} is log normally distributed, then its entropy is equal to one half of its variance. In general, entropy measures all higher order cumulants κ_i of $\log X$: $L_t(X_{t+1}) = \kappa_{2t}/2! + \kappa_{3t}/3! + \kappa_{4t}/4! + \ldots$ Similarly, following Backus, Chernov, and Boyarchenko (2016), the co-entropy is defined as $L_t(X_{t+1}Y_{t+1}) - L_t(X_{t+1}) - L_t(Y_{t+1})$, which is a natural measure of the covariation. This measure is zero if the variables are conditionally independent. Using these measures of risk, we derive an analog to Proposition 1 in the case of non-Gaussian shocks.

Proposition 2. *Under Assumption 2, when the exchange rate change is* $\Delta s_{t+1} = \eta_{t+1} + m^*_{t+1} - m_{t+1}$*, then the wedge* η *satisfies the following restrictions:*

$$E_t(\eta_{t+1}) - L_t(\exp(-\eta_{t+1})) = L_t(M_{t+1}\exp(-\eta_{t+1})) - L(M_{t+1}) - L_t(\exp(-\eta_{t+1})), \quad (14)$$
$$-E_t(\eta_{t+1}) - L_t(\exp(\eta_{t+1})) = L_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) - L\left(M^*_{t+1}\right) - L_t(\exp(\eta_{t+1})), \quad (15)$$

and where the trend $E_t(\eta_{t+1})$ *satisfies:*

$$\begin{aligned} -E_t(\eta_{t+1}) &\leq \log E_t\left(M_{t+1}\exp(-\eta_{t+1})\right) - E_t \log(M_{t+1}), \\ E_t(\eta_{t+1}) &\leq \log E_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) - E_t \log\left(M^*_{t+1}\right), \\ E_t(\eta_{t+1}) &\leq \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right) - E_t \log\left(\frac{M^*_{t+1}}{M_{t+1}}\right). \end{aligned}$$

These η-conditions are the exact equivalent of the covariance conditions in the lognormal case. When the stochastic discount factor and the wedge are jointly lognormal, as in Assumption 1, one recovers the same conditions derived in Proposition 1.

We now compare the entropy of the incomplete markets exchange rates to the entropy of the complete markets version, denoted $L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right)$, in the following corollary, a clear counterpart to Corollary 1 in the lognormal case:

Corollary 4. *Under Assumption 2, when the exchange rate change is* $\Delta s_{t+1} = \eta_{t+1} + m^*_{t+1} -$

m_{t+1}, then its entropy is:

$$
\begin{aligned}
L_t\left(\frac{S_{t+1}}{S_t}\right) &= L_t\left(\frac{M^*_{t+1}\exp(\eta_{t+1})}{M_{t+1}}\right), \\
&= L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) - E_t\left(\eta_{t+1}\right) + \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right) - \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right).
\end{aligned}
$$

Hence, the difference between the entropy of the exchange rate change in incomplete versus complete markets is equal to:

$$
\Delta L_t = L_t^{IM} - L_t^{CM} = -E_t\left(\eta_{t+1}\right) + \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right) - \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right).
$$

The change in entropy of exchange rates introduced by incomplete spanning is tightly linked to the change in currency risk premia. To see this point, let us first define the currency risk premium when shocks are non-Gaussian. Backus, Foresi, and Telmer (2001) show that the complete markets' risk premium in logs is simply $L_t\left(M_{t+1}\right) - L_t\left(M^*_{t+1}\right)$. The complete markets' risk premium in levels is thus given by:

$$
E_t[rx^{FX}_{t+1}] + L_t(S_{t+1}/S_t) = L_t\left(M_{t+1}\right) - L_t\left(M^*_{t+1}\right) + L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right).
$$

The following proposition describes the risk premium with incomplete spanning; it is the counterpart to Corollary 2.

Corollary 5. *Under Assumption 2, when the exchange rate change is $\Delta s_{t+1} = \eta_{t+1} + m^*_{t+1} - m_{t+1}$, then the risk premium in logs on a long position in foreign currency is:*

$$
E_t[rx^{FX}_{t+1}] = L_t\left(M_{t+1}\right) - L_t\left(M^*_{t+1}\right) + E_t\left(\eta_{t+1}\right).
$$

The risk premium in levels on a long position in foreign currency (from the perspective of the *domestic investor) is given by:*

$$E_t[rx^{FX}_{t+1}] + L_t(S_{t+1}/S_t) \quad = \quad L_t\left(M_{t+1}\right) - L_t\left(M^*_{t+1}\right) + E_t\left(\eta_{t+1}\right) + L_t\left(\frac{M^*_{t+1}\exp(\eta_{t+1})}{M_{t+1}}\right).$$

The difference between the currency risk premium in incomplete versus complete markets is thus related to the changes in exchange rate entropy introduced by the incomplete spanning:

$$\Delta RP_t = RP^{IM}_t - RP^{CM}_t = \Delta L_t + E_t\left(\eta_{t+1}\right).$$

In the symmetric case, where the drift of the wedge is zero ($E_t\left(\eta_{t+1}\right) = 0$), a decrease in the entropy of the exchange rate leads to a commensurate decrease in the foreign currency risk premium. When the drift of the wedge is not zero, it may be possible to lower the entropy of exchange rates without lowering their risk premia.

While Corollaries 4 and 5 offer a clear link between the entropy and risk premia of exchange rates, there is no equivalent result for their cyclicality. In a preference-free setting, we are not able to bound the co-entropy of exchange rates and stochastic discount factors. As a result, the tension that we highlight in the lognormal case cannot be formally expressed here. As entropy depends on an infinite sum of higher moments, it may be possible to pick some higher moments that affect the co-entropy of exchange rates and stochastic discount factors without affecting much the entropy of the exchange rates or the currency risk premium. We do not know of such a model, but we cannot mathematically rule out its existence.

To build some intuition, we extend our Gaussian case to a case where the SDFs and the wedge are characterized by their first three moments: mean, variance, and skewness (while the higher moments are zero). In this case, under Assumption 2, the two conditions on the moments of the wedge η (implied by the four Euler equations that characterize the domestic and foreign risk-free assets held by the domestic and foreign investors) are:

$$
\begin{aligned}
0 &= E_t(\eta_{t+1}) + \frac{1}{2}Var_t(\eta_{t+1}) + cov_t(m^*_{t+1}, \eta_{t+1}) + \frac{1}{6}Skew_t(m^*_{t+1} + \eta_{t+1}), \\
0 &= -E_t(\eta_{t+1}) + \frac{1}{2}Var_t(\eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1}) + \frac{1}{6}Skew_t(m_{t+1} - \eta_{t+1}).
\end{aligned}
$$

If the skewness is zero, we recover the same expressions as in Equations (10) and (11). Taking the skewness into account, the variance of the exchange rate is now:

$$
Var_t(\Delta s_{t+1}) = Var_t(m^*_{t+1} - m_{t+1}) - Var_t(\eta_{t+1}) - \frac{1}{3}Skew_t(m^*_{t+1} + \eta_{t+1}) - \frac{1}{3}Skew_t(m_{t+1} - \eta_{t+1}).
$$

The skewness terms are novel compared to Corollary 1. The covariance between exchange rate changes and the relative log SDFs is:

$$
covar(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}) = Var_t(\Delta s_{t+1}) + \frac{1}{6}Skew_t(m^*_{t+1} + \eta_{t+1}) + \frac{1}{6}Skew_t(m_{t+1} - \eta_{t+1}).
$$

Again, the skewness terms are novel compared to Corollary 3. If the skewness terms are positive and help decrease the volatility of the exchange rates below the complete markets' benchmark (thus helping to address the Brandt, Cochrane and Santa-Clara (2005) puzzle), they will increase the co-movement between exchange rates and relative log SDFs, thus amplifying the exchange rate disconnect puzzle. If the kurtosis of the SDFs and wedge is also nonzero, then both skewness and kurtosis will affect the volatility and cyclicality of the exchange rates. In that case, one could imagine, for example, that the kurtosis of the wedge would help reduce the volatility of exchange rates, while its skewness would decrease the exchange rate cyclicality.

In the class of non-normal models often used in the option pricing and macro-finance literature, however, we show that the tension we highlighted in the Gaussian case applies: introducing incomplete spanning to decrease the volatility of exchange rates also decreases the currency risk premium and implies counterfactual links between stochastic discount factors and exchange rate changes. Yet, since these results are model-specific, we leave them for the last section of the paper and focus for now on bringing our preference-free theoretical results to the data.

3 Quantitative Implications: Addressing Three Stylized Facts

In this section, we study the ability of incomplete spanning models to match simultaneously three facts: the volatility, the cyclicality, and the risk premium on exchange rates. We start with a brief overview of these moments in the data.

3.1 Empirical Facts

We consider 15 developed countries: Australia, Belgium, Canada, Denmark, France, Germany, Italy, Japan, Netherlands, New Zealand, Norway, Sweden, Switzerland, the U.K., and the U.S. All exchange rates are defined with respect to the U.S. dollar. Data are quarterly, over the 1973.IV – 2014.IV period. Table 1 reports some well-known stylized facts.

We start with the exchange rate volatility. Across all the countries, the average annualized volatility is 11% in this sample (Panel A of Table 1). It is precisely estimated, with a standard error (obtained by bootstrapping) of 0.4%, and there are only small variations across countries: the cross-sectional standard deviation is 1.6%. There is no statistical difference between the volatility of real and nominal exchange rates. This is the moment of the data that is the most precisely known.

We turn now to the exchange rate risk premium. In the data, as Panel B of Table 1 reports, the average carry trade excess return is 4.4%, implying a Sharpe ratio of 0.5. To obtain the estimate of the carry trade excess return, the countries are sorted by the level of their short-term nominal interest rates into four portfolios. The exchange rate risk premium corresponds to the average carry trade excess return obtained by borrowing in low-interest rate currencies (i.e., shorting the first portfolio) and investing in high-interest rate currencies (long the last portfolio). Larger average currency risk premia and Sharpe ratios can be obtained on larger sets of countries (Lustig and Verdelhan, 2007).

We end with the exchange rate cyclicality. If we again assume that SDFs are high in bad times, complete markets imply that the foreign currency appreciates in bad times for foreign investors, while the home currency depreciates in good times for domestic investors. This implication of complete markets is sometimes viewed as undesirable and counterintuitive (e.g., the

Table 1: Exchange Rate Puzzles

	Panel A: Volatility			
	Cross-country Mean	Cross-country Std	Cross-country Min	Cross-country Max
$\sigma_{\Delta s}$	11.21	1.57	6.23	12.70
	(0.44)	(0.22)	(0.56)	(0.61)
$\sigma_{\Delta q}$	11.12	1.64	6.21	12.81
	(0.44)	(0.20)	(0.48)	(0.59)
$corr(\Delta c, \Delta c^*)$	0.17	0.10	0.02	0.35
	(0.05)	(0.02)	(0.07)	(0.08)
Equity S.R.	0.22	0.12	0.00	0.48
	(0.15)	(0.04)	(0.15)	(0.21)
	Panel B: Risk Premium			
	Time-Series Mean	Time-Series Std	Time-Series Sharpe ratio	
rx_{t+1}^{FX}	4.42	8.73	0.51	
	(1.36)	(0.97)	(0.19)	
$rx_{t+1}^{FX} + \frac{1}{2}var[rx_{t+1}^{FX}]$	4.80	8.73	0.55	
	(1.32)	(0.98)	(0.19)	
$-rx_{t+1}^{FX} + \frac{1}{2}var[rx_{t+1}^{FX}]$	-4.04	8.73	-0.46	
	(1.40)	(0.98)	(0.20)	
	Cross-country Mean	Cross-country Std	Cross-country Min	Cross-country Max
β_{UIP}	-0.26	0.63	-1.28	1.10
	(0.47)	(0.17)	(0.51)	(0.60)
	Panel C: Cyclicality			
	Cross-country Mean	Cross-country Std	Cross-country Min	Cross-country Max
$corr(\Delta q, \Delta c - \Delta c^*)$	-0.07	0.09	-0.22	0.14
	(0.05)	(0.03)	(0.07)	(0.10)
$\beta_{Backus-Smith}$	-0.01	0.02	-0.03	0.02
	(0.01)	(0.00)	(0.01)	(0.02)
$corr(-\Delta q, \Delta c^*)$	-0.02	0.12	-0.21	0.24
	(0.03)	(0.03)	(0.07)	(0.09)

Notes: The table reports summary statistics on three exchange rate puzzles. Panel A focuses on the exchange rate volatility. It reports the cross-country mean of the bilateral nominal and real exchange rate volatilities, along with the cross-country standard deviation of the bilateral exchange rate volatilities and the corresponding minimum and maximum values across countries. Panel A also reports similar moments for the correlation between U.S. and foreign consumption growth rates and equity Sharpe ratios on MSCI country indices. Panel B focuses on the exchange rate risk premium. It reports the time-series mean carry trade excess return, its time-series standard deviation and its Sharpe ratio (obtained as the ratio of the mean excess return to its standard deviation). The excess returns are either in logs, or in levels, from the perspective of the U.S. or foreign investor. Finally, Panel B reports the slope coefficient in a regression of exchange rate changes on the foreign minus domestic interest rate difference. Excess returns are annualized (multiplied by 4) and reported in percentages. The standard deviation on the carry trade returns in annualized (multiplied by 2) and reported in percentages. The countries are sorted by the level of their short-term nominal interest rates into four portfolios. The exchange rate risk premium corresponds to the average carry trade excess return obtained by borrowing in low-interest rate currencies (i.e., shorting the first portfolio) and investing in high-interest rate currencies (long the last portfolio). Panel C focuses on the exchange rate cyclicality. It reports similar moments for the correlation between the changes in real exchange rates and the relative consumption growth, the slope coefficient in a regression of relative consumption growth rates on exchange rate changes and a constant, and for the correlation between the changes in real exchange rates and the foreign consumption growth. Data are quarterly, over the 1973.IV – 2014.IV period. The panel consists of 15 countries: Australia, Belgium, Canada, Denmark, France, Germany, Italy, Japan, Netherlands, New Zealand, Norway, Sweden, Switzerland, U.K., and U.S. The standard errors (reported between brackets) were generated by block-bootstrapping 10,000 samples, each block containing 2 quarters.

Argentine peso depreciated in 2002, clearly not during great times in Argentina), although not always rejected by the data (e.g., the Japanese yen appreciated after the 2014 tsunami in Japan). To be more precise requires a model of the SDF and its drivers. Historically, a large literature assumes that the SDFs are driven by consumption growth shocks. In the data, as shown in Panel C of Table 1, the corresponding unconditional correlation between exchange rate changes and relative consumption growth rates is not statistically different from zero. This is the Kollmann (1991) and Backus and Smith (1993) puzzle. Likewise, the unconditional correlation between changes in exchange rates and foreign consumption growth or the slope coefficient in a regression of relative consumption growth rates on exchange rate changes and a constant are also not statistically different from zero. Since exchange rate changes do not seem to comove significantly with any macroeconomic variables, it seems fair to assume that the cyclicality of exchange rates is empirically zero.

To sum up, we focus on three facts about exchange rates: (i) a standard deviation of 11%, (ii) a currency risk premium of around 4%, and (iii) a cyclicality close to zero. The high equity Sharpe ratio imposes some restrictions on the volatility of SDFs.

3.2 Volatile SDFs

To study the ability of incomplete spanning models to match quantitatively these three facts, we need to pin down some moments of the SDFs. We make one additional assumption.

Assumption 3. *We assume that the log domestic and foreign stochastic discount factors, m and m^*, are volatile enough to match the equity risk premium.*

Recall that the standard deviation of the SDF measures the maximum Sharpe ratio in a lognormal world. We assume that the maximum Sharpe ratio is 0.5, in line with the equity premium in the U.S. over a long sample and the currency risk premium in our sample. A maximum Sharpe ratio of 0.50 is a conservative estimate: while MSCI indices that track the unconditional returns on large firms exhibit relatively low Sharpe ratios, many investment strategies (e.g. conditional on firms' characteristics or using different asset classes) deliver Sharpe ratios well beyond 0.5. Many hedge funds would claim to attain Sharpe ratios above one. We maintain our conservative estimate of the maximum Sharpe ratio, as any larger values further raise the bar for incomplete spanning models. In the main text, we assume that the home and foreign exhibit the same volatility (0.5) but consider different volatilities in the appendix.

We consider all possible cross-country correlations between the log domestic and foreign stochastic discount factors. In the data, the correlation of consumption growth rates varies between 0.02 and 0.35 as shown in Panel A of Table 1. In this respect, a cross-country correlation of 0.5 between m and m^* appears high. Yet, to test the robustness of our results, we consider a range of correlation, from 0 to to 0.99.

With these targets in mind, we turn now to the implied exchange rate moments across incomplete market models.

3.3 Exchange Rate Volatility

We want to match a 11% per annum volatility of exchange rate changes. Corollary 1 implies that we can simply back out the volatility of the wedge needed to match the volatility of exchange rates in the data:

$$var_t(\eta_{t+1}) \quad = \quad var_t(m_{t+1}) + var_t(m^*_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1}) - 0.11^2.$$

This equals the difference between the variance of the complete market exchange rates implied by the stochastic discount factors and the target variance. Figure 1 plots the necessary volatility of the wedge to match the actual exchange rate volatility for different values of the cross-country correlation of the SDFs. For example, under Assumption 3 and with a cross-country correlation of 0.5, the wedge must have a standard deviation of 49% per annum ($std_t(\eta_{t+1}) = 0.49$), close to the maximum Sharpe ratio. If the cross-country correlation of the SDFs decreases from 0.5 to zero, then one needs an even more volatile wedge: $std_t(\eta_{t+1})$ increases to 70%. When the cross-country correlation is 0.99, there is no need to introduce any wedge to match the volatility of exchange rate. But for any cross-country correlation of SDFs below 0.8, the wedge has to exhibit a volatility of at least 30%.

As Figure 1 shows, the results do not change much with the targeted volatility: the conclusion

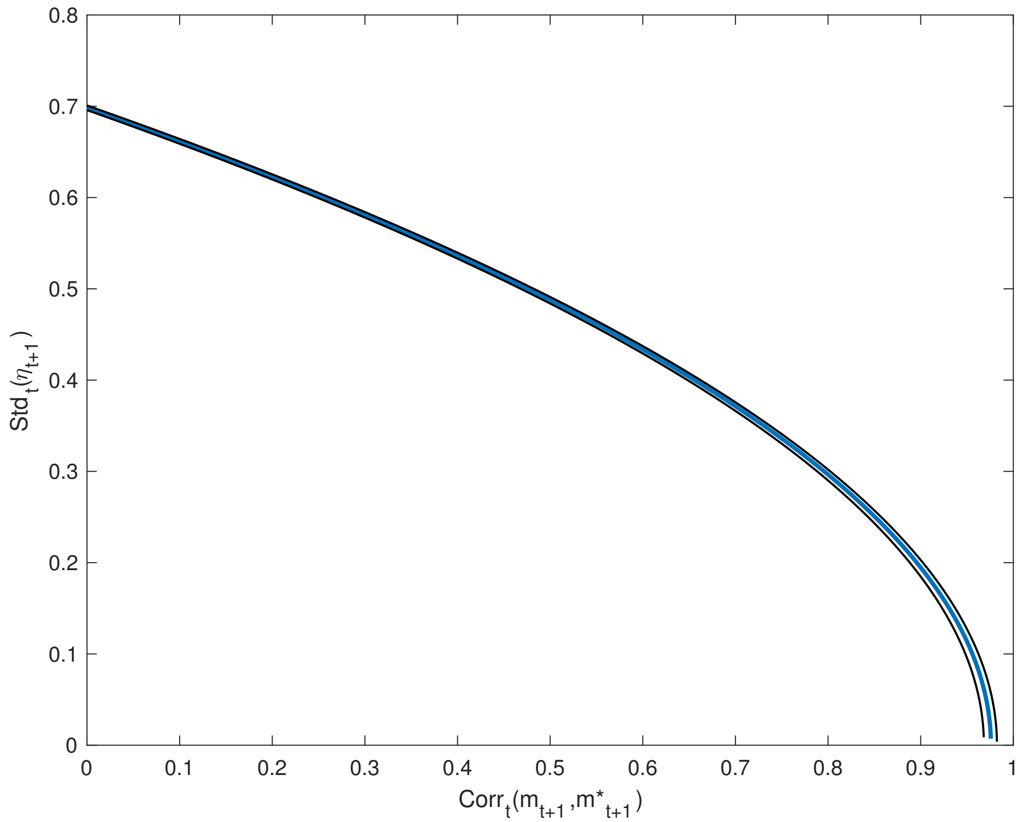

Figure 1: Matching Exchange Rate Volatility: The Volatility of the Wedge vs the Cross-country Correlation of SDFs — The figure reports the volatility of wedge, denoted $std_t(\eta_{t+1})$, that is needed to match the volatility of the changes in the log exchange rate, against the cross-country correlation of SDFs, denoted $\rho_t(m_{t+1}, m^*_{t+1})$. The figure is drawn assuming a maximum Sharpe ratio of 0.50 in both countries ($std_t(m_{t+1}) = 0.50$ and $std_t(m^*_{t+1}) = 0.50$). The average volatility of exchange rates, $std_t(\Delta s_{t+1})$, in our sample is 11% (blue line). The figure also plots (black lines) the same relationship for exchange rate volatilities one cross-country standard deviation (1.6%) above and below the cross-country mean volatility (11%).

is similar for targets that are one standard deviation above or below the mean exchange rate volatility. The figure is drawn assuming a maximum (annualized) Sharpe ratio of 0.50. For higher values, the volatility of the wedge increases. In a nutshell, incomplete spanning helps with the volatility puzzle, but for the usual macroeconomic cross-country correlations, the volatility of the wedges is of the same order of magnitude as the maximum Sharpe ratio. The first moment of the wedge is here irrelevant.

3.4 Currency Risk Premia

We turn now to the currency risk premium. Once we match the exchange rate volatility, the currency risk premium no longer depends on the correlation between the home and foreign stochastic discount factors. As already noted, the log currency risk premium in complete markets is simply $1/2\left[var_t\left(m_{t+1}\right) - var_t\left(m^*_{t+1}\right)\right]$. Under Assumptions 1 and 2, when markets are incomplete, the currency risk premia in logs as well as in levels, from the perspective of the home and foreign investors, are:

$$
\begin{aligned}
E_t[rx^{FX}_{t+1}] &= \frac{1}{2}\left[var_t\left(m_{t+1}\right) - var_t\left(m^*_{t+1}\right)\right] + E_t\left(\eta_{t+1}\right). \\
E_t[rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] &= \frac{1}{2}var_t(\Delta s_{t+1}) + \frac{1}{2}\left[var_t\left(m_{t+1}\right) - var_t(m^*_{t+1})\right] + E_t\left(\eta_{t+1}\right) \\
E_t[-rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] &= \frac{1}{2}var_t(\Delta s_{t+1}) - \frac{1}{2}\left[var_t\left(m_{t+1}\right) - var_t(m^*_{t+1})\right] - E_t\left(\eta_{t+1}\right).
\end{aligned}
$$

Figure 2 plots the theoretical currency risk premium in logs and levels and its empirical counterpart. The parameters are identical to those in Figure 1, matching an exchange rate volatility of 11%. The currency risk premia are plotted against the first moment of the wedge, $E_t(\eta_{t+1})$, ranging from -5% to 5%.

First, when the disturbance η is mean-zero ($E_t\left(\eta_{t+1}\right) = 0$), incomplete spanning does not introduce any non-predictability in exchange rates. In this case, the effects of the η perturbations are completely identical for the home and foreign countries. This is a natural benchmark case to consider. In this case, the risk premia, in logs as in levels, are the same as in complete markets. In a symmetric model, where the volatilities of the home and foreign SDFs are the same, the log currency risk premium is zero, and the currency risk premia in levels are small, equal to half of the exchange rate variance. Without introducing exchange rate predictability, such incomplete spanning models that match the exchange rate volatility cannot match the actual currency risk premia.

Then, when the drift of the wedge increases, the theoretical currency risk premium tends towards its empirical counterpart. For a wedge of 5%, the theoretical currency risk premium matches the actual value. Such a drift, however, implies that exchange rate changes are pre-

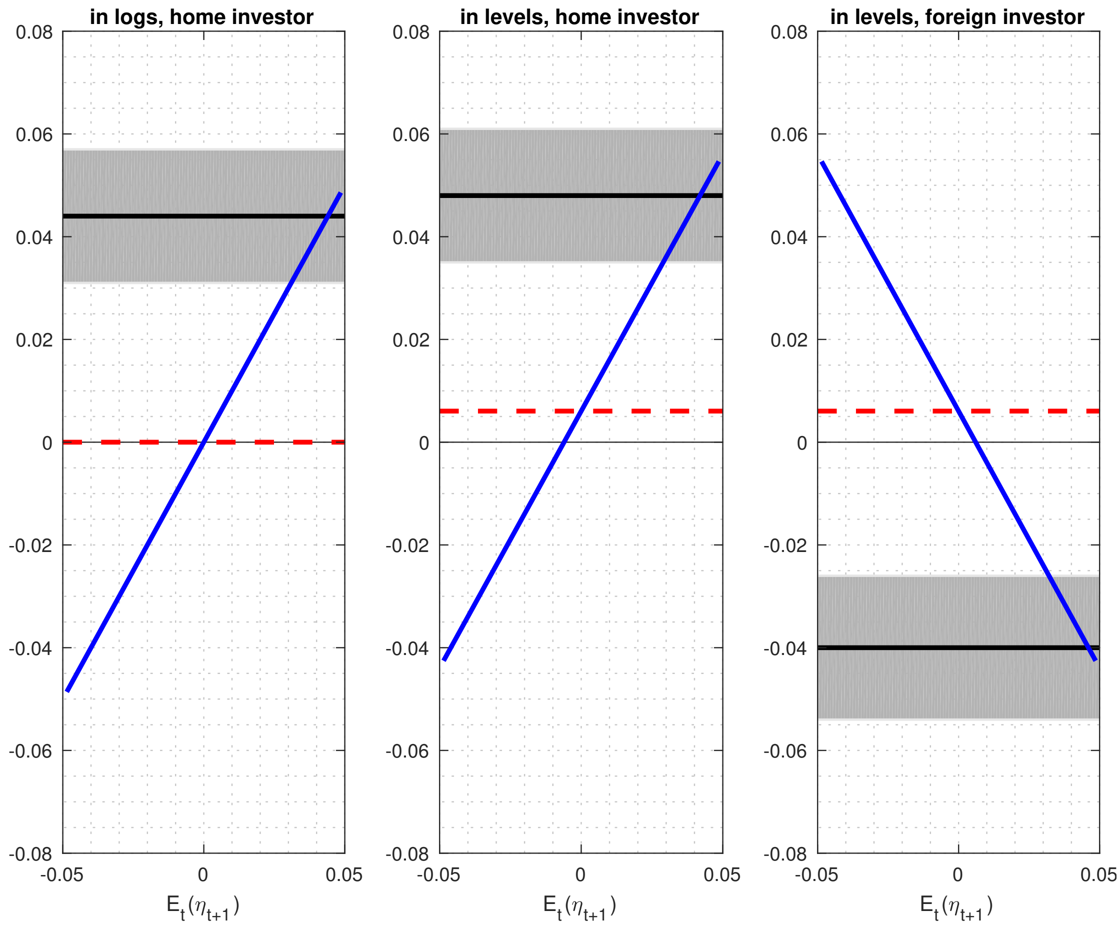

Figure 2: Currency Risk Premia: The figure reports the foreign currency risk premium in logs (left panel), as well as in levels, from the perspective of the home investor (center panel) or foreign investor (right panel), against the first moment of the incomplete market wedge, denoted $E_t(\eta_{t+1})$. The figure is drawn assuming a maximum Sharpe ratio of 0.50 in both countries ($std_t(m_{t+1}) = 0.50$ and $std_t(m^*_{t+1}) = 0.50$). The volatility of the wedge, $std_t(\eta_{t+1})$, is chosen to match the empirical volatility of the exchange rate changes (11%). The red dotted line shows each moment in a complete market model with the same SDF volatilities. The gray area indicates the value of the average carry trade excess return in the data: it is centered around the mean log excess return (4.4%, left panel) or the mean excess return from the perspective of the home and foreign investor (4.8% and −4.0% in the center and right panels); the area represents one standard error (1.3%) above and below the mean.

dictable, while in the data, exchange rates are hard to predict. Among developed countries over the post-Bretton Woods period, exchange rates look like random walk without drift. Thus, how big could the drift of the wedge be?

Recent theoretical and empirical work in asset pricing sheds some light on this question. In theory, the expected changes in exchange rates come from the expected difference in stochastic discount factors and the expected value of the wedge. Alvarez and Jermann (2005) decompose the pricing kernel in a martingale component (denoted M^P_{t+1}) and a transitory component (denoted M^T_{t+1}): $M_{t+1} = M^P_{t+1} M^T_{t+1}$, where $E_t(M^P_{t+1}) = 1$ and M^T_{t+1} is the inverse of the holding period return on an infinite maturity bond. The drift of the wedge is then equal to the cross-country difference in holding period returns of infinite maturity bonds (hpr^{∞}_{t+1}) once converted in the same units, and the difference in entropy of the martingale components of the SDF:[5]

$$E_t[\eta_{t+1}] = E_t[\Delta s_{t+1}] + E_t[hpr^{*,\infty}_{t+1}] - E_t[hpr^{\infty}_{t+1}] + L_t(M^{P*}_{t+1}) - L_t(M^P_{t+1})$$

When the last two terms cancel out, for example when permanents shocks are the same across countries ($M^{P*}_{t+1} = M^P_{t+1}$), then the drift of the wedge is pinned down by the difference in holding period returns. Using 10-year bonds as proxies for the long-term bonds, Lustig et al. (2017) find that this difference is not statistically significant post-Bretton Woods among developed countries. Thus, in this case, the drift should be zero.[6]

As we shall see, introducing a positive drift, while boosting risk premia, does not help with exchange rate cyclicality.

3.5 Exchange Rate Cyclicality

Figure 3 plots different measures of exchange rate cyclicality against the drift of the wedge. The parameters are the same as for Figures 1 and 2, where the volatility of the wedge is chosen to

[5] Start from the definition of the wedge in Equation (3) and introduce the decomposition of the SDF:

$$\begin{aligned} E_t[\Delta s_{t+1}] &= E_t[\eta_{t+1}] + E_t[m^*_{t+1}] - E_t[m_{t+1}], \\ &= E_t[\eta_{t+1}] + E_t[m^{P*}_{t+1}] - E_t[m^P_{t+1}] + E_t[m^{T*}_{t+1}] - E_t[m^T_{t+1}]. \end{aligned}$$

As usual, lower letters denote logs. Since the permanent component is a martingale, then $E_t(m^P_{t+1}) = -L_t(M^P_{t+1})$, where L_t denotes the entropy of a variable ($L_t(M^P_{t+1}) = 0.5Var_t(m^P_{t+1})$ in the case of a Gaussian SDF). By construction of the SDF decomposition, $E_t[m^T_{t+1}] = -E_t[hpr^{\infty}_{t+1}]$.

[6] Overall, either long-term bond returns are more predictable than we know, or the drift in the wedge has to be equal to the risk premium of the permanent shocks. For example, a risk premium of 8% in the U.S. and 13% in a foreign country, if purely driven by permanent shocks to the SDFs, would entail a drift of 5%. In the data, average aggregate equity returns tend to be higher in the U.S. than in other developed countries, suggesting that the drift may be close to 0 if not negative.

match the volatility of the exchange rate changes.

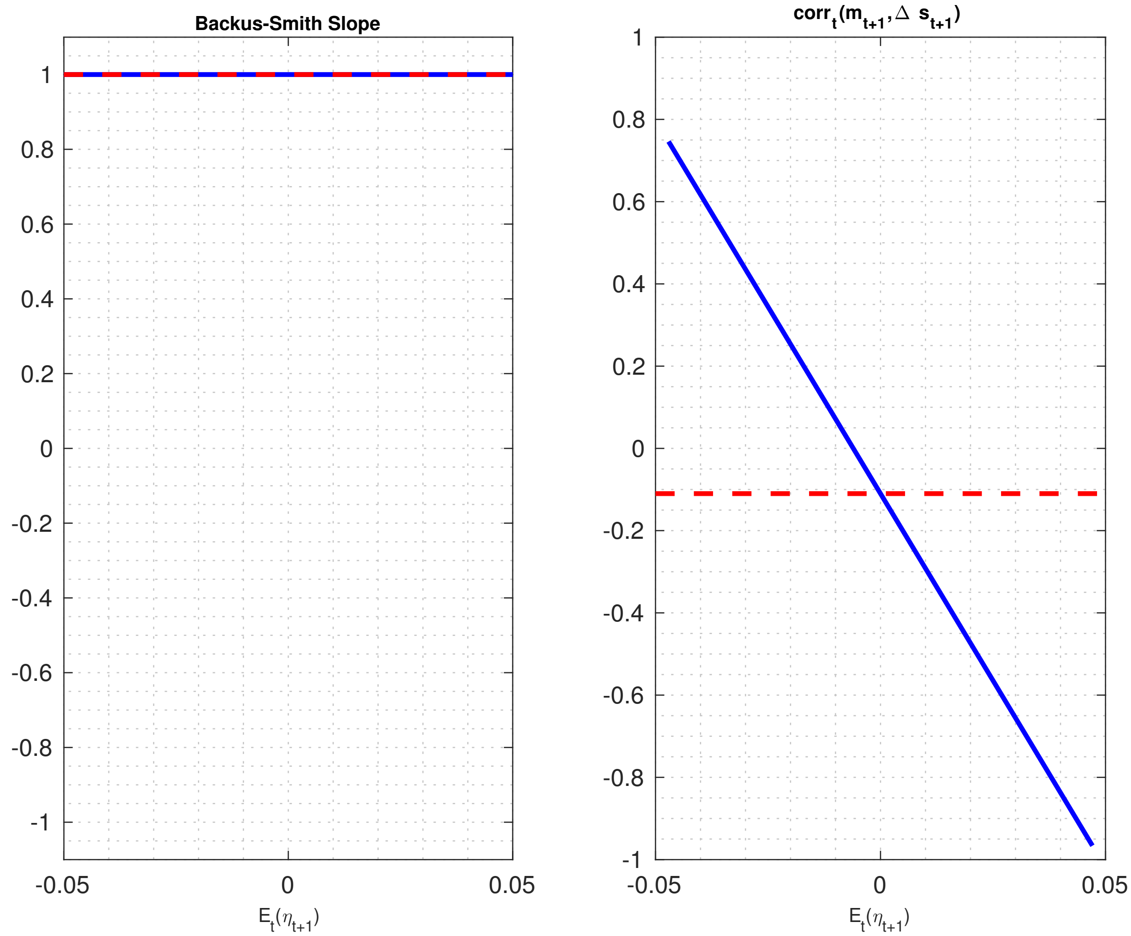

Figure 3: Exchange Rate Cyclicality: The figure reports the slope coefficient in a regression of the difference in log SDFs, $m^*_{t+1} - m_{t+1}$ on the log change in exchange rates (left panel) and the correlation between the log home SDF and the change in the exchange rates, $corr_t(\Delta s_{t+1}, m_{t+1})$, (left panel) against the first moment of the incomplete market wedge, denoted $E_t(\eta_{t+1})$. The red dotted line shows the values of these three moments when markets are complete. The figure is drawn assuming a maximum Sharpe ratio of 0.50 in both countries ($std_t(m_{t+1}) = 0.50$ and $std_t(m^*_{t+1}) = 0.50$).

The left panel reports the Backus-Smith slope obtained in a regression of exchange rate changes on the difference between the foreign and home SDFs. As shown in Corollary 1, the Backus-Smith slope is the same in incomplete and complete markets under Assumptions 1 and 2, and thus equal to one. Incomplete spanning models do not help addressing the Backus-Smith puzzle when agents trade risk-free bonds in a lognormal world.

The right panel reports the correlation between the log home SDF and the change in the exchange rates, $corr_t(\Delta s_{t+1}, m_{t+1})$. In complete markets, as already noted, the home currency appreciates when the home country experiences difficult times, as encoded in a large SDF: $corr_t(\Delta s_{t+1}, m_{t+1})$ is thus negative. For large values of the drift that help match the currency risk premium, incomplete spanning models reinforce this unappealing feature: $corr_t(\Delta s_{t+1}, m_{t+1})$ is even more negative than in complete market models. In this case, the incomplete market model is even less attractive than its complete market counterpart.

The tight link between exchange rate cyclicality and currency risk premia is independent from the cross-country correlation of SDFs, as shown below:

$$corr_t\left(m_{t+1}, \Delta s_{t+1}\right) = \frac{covar_t\left(m_{t+1}, \Delta s_{t+1}\right)}{std_t\left(m_{t+1}\right) std_t\left(\Delta s_{t+1}\right)} = -\frac{E_t[rx_{t+1}^{FX}] + \frac{1}{2} var_t[rx_{t+1}^{FX}]}{std_t\left(m_{t+1}\right) std_t\left(\Delta s_{t+1}\right)} = \frac{-SR_t^{FX}}{std_t\left(m_{t+1}\right)},$$

where SR^{FX} denotes the Sharpe ratio on the currency risk premium. Note that this expression is valid in complete and incomplete markets. As we lower the correlation between exchange rates and the home stochastic discount factor, we also lower the currency Sharpe ratios proportionally. When the correlation is zero, the risk premium is zero. To change the sign of this correlation, one would need to write a model with negative currency risk premia.

Overall, incomplete spanning models where agents trade risk-free bonds in a lognormal world may help matching the volatility of exchange rate changes, but they then only increase the exchange rate risk premia at the cost of introducing exchange rate predictability, and they do not improve the exchange rate cyclicality. To illustrate these general results, we turn now to some specific models.

4 Model-Specific Examples

In this section, we first consider some model-specific examples of our general framework, from the simple consumption-CAPM model to its extension to jumps and to dynamic asset pricing models. We end the paper with a brief review of the existing incomplete market models of exchange rates as seen through the lenses of our general results.

4.1 A Simple Consumption-CAPM Example

In the tradition of Lucas (1982), consider a model in which the domestic and foreign representative agents have power utility with risk aversion coefficient γ. The domestic aggregate consumption growth Δc consists of a standard Gaussian component $w \sim N(\mu, \sigma^2)$. The same applies to foreign consumption growth: $w^* \sim N(\mu^*, \sigma^{2,*})$. The domestic and foreign consumption growth rates are thus:

$$\Delta c_{t+1} = w_{t+1}, \tag{16}$$

$$\Delta c^*_{t+1} = w^*_{t+1}, \tag{17}$$

where the correlation of domestic and foreign shocks is ρ_{w,w^*}. Assume that the incomplete market wedge takes the form: $\eta_{t+1} = \gamma d_{t+1}$, where $d \sim N(\mu_d, \sigma_d^2)$. The correlations between the initial consumption growth shocks and the wedge are denoted $\rho_{w,d}$ and $\rho_{w^*,d}$. In this case, Proposition 1 implies that the wedges satisfy:

$$\mu_d = \gamma^2\sigma_d^2/2 + \rho_{w,d}\gamma^2\sigma\sigma_d, \tag{18}$$

$$-\mu_d = \gamma^2\sigma_d^2/2 - \rho_{w^*,d}\gamma^2\sigma^*\sigma_d. \tag{19}$$

With only two consumption growth innovations, the only interesting case is as follows: the domestic investor cannot invest in any foreign risky asset. If the foreign investor could invest in more assets, then the additional covariance restrictions in Equation (13) would apply, and markets would be complete.[7] We thus focus on the case where investors can only invest in risk-free bonds.

Volatility In the absence of wedges, the volatility of the exchange rate changes is $\gamma^2\sigma^{*2} + \gamma^2\sigma^2 - 2\gamma^2\rho_{w,w^*}\sigma\sigma^*$. Adding the wedge, as shown in Corollary 1, reduces the exchange rate

[7]These conditions imply that d_{t+1} is orthogonal to w_{t+1} and w^*_{t+1}, because the log return on the domestic (foreign) risky asset is affine in the domestic (foreign) innovation. The additional covariance restrictions in Equation (13), $\rho_{w,d} = 0$ and $\rho_{w^*,d} = 0$, combined with Equations (18) and (19), imply that $\sigma_d = 0$ and $\mu_d = 0$. We are back in the case of complete markets: $\eta_{t+1} = 0$.

variance by $\gamma^2\sigma_d^2$. The unspanned shocks d_{t+1} are counter-cyclical, i.e. negatively correlated with domestic consumption growth, and as a result, always reduces the exchange rate's volatility.

Cyclicality The Backus-Smith correlation coefficient is given by:

$$1 \geq corr_t\left(\Delta c_{t+1} - \Delta c^*_{t+1}, \Delta s_{t+1}\right) = \frac{\sqrt{\gamma^2\sigma^{*,2} + \gamma^2\sigma^2 - 2\gamma^2\rho_{w,w^*}\sigma\sigma^* - (\gamma\sigma_d)^2}}{\sqrt{\gamma^2\sigma^{*,2} + \gamma^2\sigma^2 - 2\gamma^2\rho_{w,w^*}\sigma\sigma^*}} \geq 0$$

The correlation is smaller than one, its complete markets value, but always positive. In the symmetric case where the drift μ_d is zero, the unspanned risk is always negatively correlated with domestic consumption growth, as implied by Equation (18), and positively correlated with foreign consumption growth, as implied by Equation (19).

Risk Premium When markets are complete, the currency risk premium in levels (defined from the perspective of the home investor) is given by $\gamma^2\sigma^2 - \gamma^2\rho_{w,w^*}\sigma\sigma^*$. Likewise, the currency risk premium in levels, this time defined from the perspective of the foreign investor, is equal to $\gamma^2\sigma^{*2} - \gamma^2\rho_{w,w^*}\sigma\sigma^*$. When markets are incomplete, the currency risk premia change. The difference in the currency risk premium (defined from the perspective of the home investor) between the incomplete and complete market cases, $\Delta RP = RP_t^{IM} - RP_t^{CM}$, is equal to $\Delta RP = \rho_{w,d}\gamma^2\sigma\sigma_d$. Similarly, the difference in the currency risk premium (defined from the perspective of the foreign investor) is $\Delta RP^* = -\rho_{w^*,d}\gamma^2\sigma\sigma_d$. Hence, the total change in the risk premia has to be negative; Equations (18) and (19) imply that the sum of the last two expressions is negative. In simple words, a positive drift to mitigate the effect on currency risk premia from the perspective of the domestic investor implies a larger decline for the other investor. The Lucas (1982) model provides a simple example to our preference-free results. We now extend the model to include jumps.

4.2 A Consumption-CAPM with Jumps

As before, the domestic and foreign representative agents have power utility with identical risk aversion γ. But consumption growth in each country consists of a standard Gaussian component and a jump component. The first component is the same as in the previous consumption-based example; it is denoted w and normally distributed as $N(\mu, \sigma^2)$. The second component is a Poisson mixture of normals, denoted z. Foreign variables are denoted with a *. Log consumption growth is the sum of these two components:

$$\Delta c_{t+1} = w_{t+1} + z_{t+1}, \tag{20}$$

$$\Delta c^*_{t+1} = w^*_{t+1} + z^*_{t+1}. \tag{21}$$

At each date, the number of jumps j takes on non-negative integer values with probabilities $e^{-\varpi}\varpi^j/j!$. The parameter ϖ, the jump intensity, is the mean of j. Each jump triggers a draw from a normal distribution with mean θ and variance δ^2 for the domestic agent and with mean θ^* and variance δ^{*2} for the foreign agent. The jumps are thus common across countries, but the jump sizes are not. Conditional on the number of jumps j, the domestic jump component is normally distributed as $z_t|j \sim N(j\theta, j\delta^2)$, while the foreign jump component is normally distributed as $z^*_t|j \sim N(j\theta^*, j\delta^{2,*})$. If ϖ is small, the jump model is well approximated by a Bernoulli mixture of normals. If ϖ is large, multiple jumps can occur frequently. This functional form is known as the Merton (1976) model. In the macro-finance literature, it has been applied notably by Bates (1988), Naik and Lee (1990), Backus, Chernov, and Zin (2011), and Martin (2013).

Next, we introduce incomplete spanning in this model. We assume that the wedge takes the form $\eta_{t+1} = \gamma d_{t+1}$, where d_{t+1} follows the same Poisson mixture as z_{t+1}, but with parameters θ_d and δ_d. Conditional on the number of jumps j, the jump and wedge components are jointly normal: $z_t|j \sim N(j\theta, j\delta^2)$ and $d_t|j \sim N(j\theta_d, j\delta_d^2)$. We use $\rho_{z,d}$ and $\rho_{z^*,d}$ to denote the correlation of jump sizes between the spanned and unspanned components of exchange rates. The jumps are common for the z_{t+1} and d_{t+1} components.

Result 4. *Following Proposition 2, the wedges satisfy the following restrictions:*

$$-\gamma\theta_d + \gamma^2\delta\delta_d\rho_{z,d} + \frac{\gamma^2\delta_d^2}{2} = 0, \tag{22}$$

$$\gamma\theta_d - \gamma^2\delta^*\delta_d\rho_{z^*,d} + \frac{\gamma^2\delta_d^2}{2} = 0. \tag{23}$$

Corollary 4 implies that the change in volatility from complete to incomplete spanning is given by:

$$\Delta L_t = L_t^{IM} - L_t^{CM} = -\gamma\varpi\theta_d + \varpi e^{-\gamma\theta^* + \gamma\theta - \gamma^2\rho_{z,z^*}\delta\delta^* + (\gamma\delta)^2/2 + (\gamma\delta^*)^2/2}\left(e^{\gamma^2\delta\delta_d\rho_{z,d}} - 1\right). \tag{24}$$

When markets are complete, the foreign currency risk premium in levels (from the perspective of the domestic investor) is given by:

$$\begin{aligned} E_t\left[rx_{t+1}^{FX}\right] + L_t\left[rx_{t+1}^{FX}\right] &= \gamma^2\sigma^2 + \varpi\left(e^{-\gamma\theta + (\gamma\delta)^2/2} - 1\right) - \varpi\left(e^{-\gamma\theta^* + (\gamma\delta^*)^2/2} - 1\right) \\ &+ \varpi\left(e^{-\gamma\theta^* + \gamma\theta - 2\gamma^2\rho_{z,z^*}\delta\delta^* + (\gamma\delta)^2/2 + (\gamma\delta^*)^2/2} - 1\right). \end{aligned}$$

Introducing incomplete spanning wedges, Corollary 5 implies that the corresponding change in the risk premium is given by:

$$\Delta RP_t = RP_t^{IM} - RP_t^{CM} = \varpi e^{-\gamma\theta^* + \gamma\theta - \gamma^2\rho_{z,z^*}\delta\delta^* + (\gamma\delta)^2/2 + (\gamma\delta^*)^2/2}\left(e^{\gamma^2\delta\delta_d\rho_{z,d}} - 1\right). \tag{25}$$

When the wedge does not have a drift ($\theta_d = 0$), Equations (24) and (25) imply that, again, the market incompleteness change the exchange rate volatility and the exchange rate risk premium by the same amount. More precisely, in the absence of a drift, the market incompleteness always reduces the exchange rate volatility and the exchange rate risk premium. Equations (22) and (23) imply that the correlation is given by $\rho_{z,d} = -\rho_{z^*,d} = -0.5\delta_d/\delta$. The change in volatility and risk premium is thus negative:

$$\Delta RP_t = RP_t^{IM} - RP_t^{CM} = \varpi e^{-\gamma\theta^* + \gamma\theta - \gamma^2 \rho_{z,z^*}\delta\delta^* + (\gamma\delta)^2/2 + (\gamma\delta^*)^2/2} \left(e^{-\frac{\gamma^2 \delta_d^2}{2}} - 1 \right) = \Delta L_t < 0.$$

We turn to a simple calibration, where the wedge does not have a drift ($\theta_d = 0$) and countries are symmetric ($\theta = \theta^*$, $\delta = \delta^*$) in order to study the magnitudes of volatilities and risk premia.

Calibration We follow Backus, Chernov, and Martin (2011) and set the risk-aversion parameter (γ) to 5.19, the mean (μ) and standard deviation (σ) of the normal consumption growth shocks to 2.3% and 1%, the jump intensity ϖ to 1.7%, the mean jump size θ to -38%, and the jump size volatility δ to 25%. These parameters were chosen to match the international evidence reported in Nakamura, Steinsson, Barro, and Ursua (2013). We assume that the jump sizes are uncorrelated across countries, but the jumps are common. The absence of idiosyncratic jumps helps the model to generate low exchange rate volatility.

Figure 4 plots the exchange rate volatility $\sqrt{2L}$ and the currency risk premium on a long position in foreign currency from the perspective of the home investor. In this calibration, there is no wedge that can simultaneously deliver a reasonable exchange rate volatility and a significant risk premium. When the variance of the jumps in the wedge reaches its maximum, the exchange rate volatility is still close to 20% and the currency risk premium is less than 2%.

Finally, we also explored a calibration due to Backus, Chernov, and Martin (2011) that is based on equity index options rather than aggregate consumption growth data with more frequent but much smaller jumps. In this Merton model, we choose γ =8.70 , $\sigma = 2.53\%$, $\omega = 139\%$, $\theta = -0.74\%$ and $\delta = 1.91\%$. These results are displayed in Figure 5, which, again, plots the exchange rate volatility $\sqrt{2L}$ and the currency risk premium on a long position in foreign currency. When we use the more conservative calibration with smaller, more frequent disasters, we can match the exchange rate volatility, but the currency risk premia are too small. In addition, varying the coefficient of risk aversion does not resolve this tension. Since we cannot even match these two exchange rate moments in a rare disaster model, at least not in a model with zero drift in the wedges, we ignore the exchange rate cyclicality puzzle. The next section illustrates why the drift of the exchange rate wedge is not really a free parameter in a large class of dynamic asset pricing models.

4.3 Dynamic Asset Pricing Models

We now extend our benchmark results to a large class of dynamic asset pricing models. Specifying a law of motion for the stochastic discount factor further restrains the ability of the incomplete spanning wedge to address the main currency puzzles because it completely pins down the first moment of the wedge. We use the Cox, Ingersoll, and Ross (1985) model (denoted CIR) model to illustrate this finding. Similar results appear naturally in the case of CRRA preferences with heteroskedastic consumption, since that model is isomorphic to the CIR model. For the sake of clarity and space, we focus on a simple CIR model with country-specific factors. The Appendix presents a CIR model with common factors and a consumption-based model.

In discrete time, the simplest version of the CIR model is defined by the following two equations:

$$\begin{aligned} -\log M_{t+1} &= \alpha + \chi z_t + \sqrt{\gamma z_t} u_{t+1}, \\ z_{t+1} &= (1-\phi)\theta + \phi z_t - \sigma\sqrt{z_t} u_{t+1}, \end{aligned}$$

where M denotes the home stochastic discount factor. The disturbances $u_{t+1} \sim \mathbb{N}(0,1)$ are i.i.d. over time.[8] The foreign stochastic discount factor follows a similar law of motion but with its

[8] In this model, log bond prices are affine in the state variable z_t: $p_t^{(n)} = -B_0^n - B_1^n z_t$. The price of a one period-bond is: $P^{(1)} = E_t\left(M_{t+1}\right) = e^{-\alpha-(\chi-\frac{1}{2}\gamma)z_t}$. Bond prices are defined recursively by the Euler equation: $P_t^{(n)} = E_t(M_{t+1}P_{t+1}^{(n-1)})$. Thus the bond price coefficients evolve according to the following second-order difference equations:

$$\begin{aligned} B_0^n &= \alpha + B_0^{n-1} + B_1^{n-1}(1-\phi)\theta, \\ B_1^n &= \chi - \frac{1}{2}\gamma + B_1^{n-1}\phi - \frac{1}{2}\left(B_1^{n-1}\right)^2\sigma^2 + \sigma\sqrt{\gamma}B_1^{n-1}. \end{aligned}$$

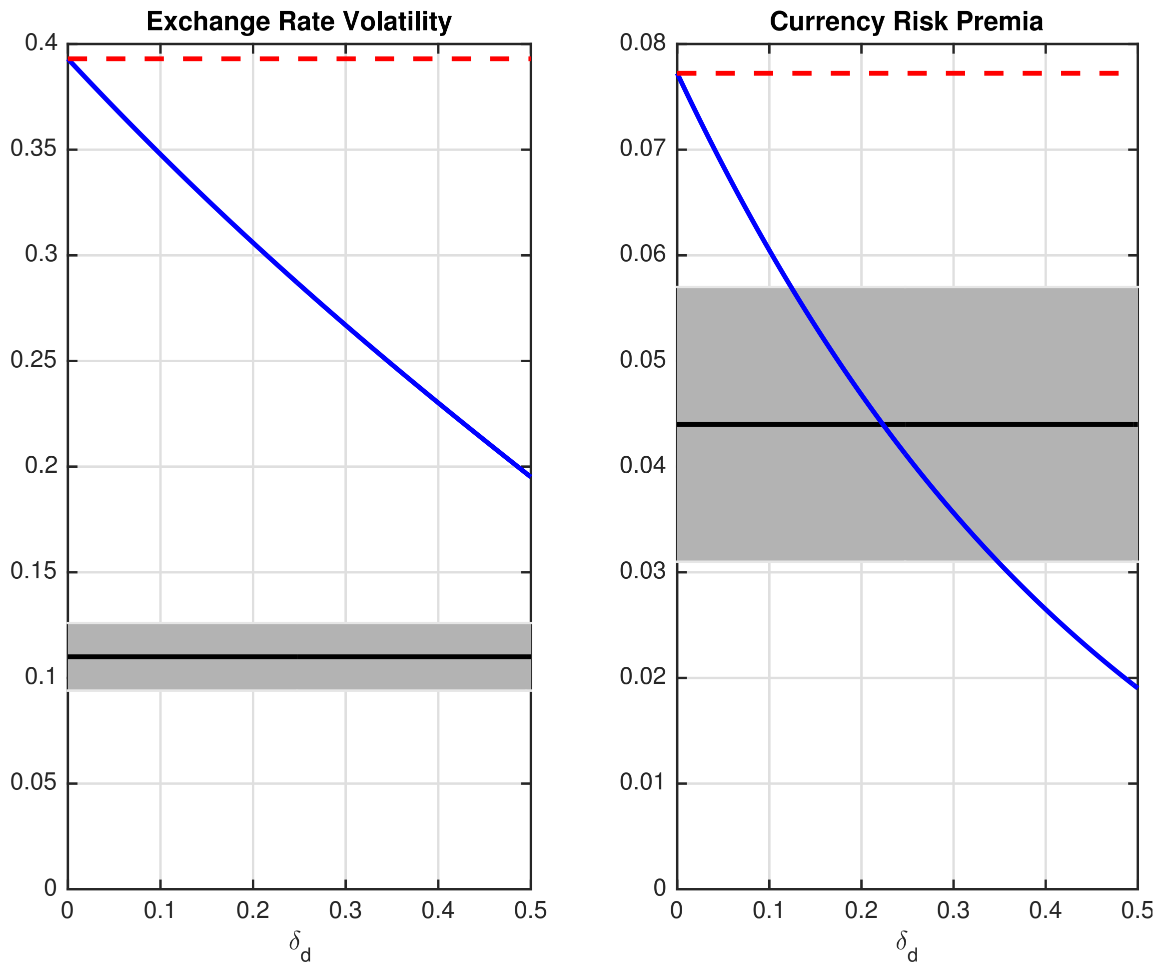

Figure 4: Exchange Rate Volatility and Risk Premia in a Barro-Rietz disaster version of the Merton (1976) Model: In the model, the incomplete spanning wedge follows the same Poisson mixture as the jump components of consumption growth. Each jump triggers a draw from a normal distribution with mean θ and variance δ^2 for the domestic agent, with mean θ^* and variance δ^{*2} for the foreign agent, and with mean θ_d and variance δ_d^2 for the wedge. We assume that the wedge does not introduce non-stationarity in exchange rates ($\theta_d = 0$), that the two countries' consumption growth processes are symmetric ($\theta = \theta^*$, $\delta = \delta^*$) and that the jump components are not correlated across countries. We follow Backus, Chernov, and Martin (2011) and set the risk aversion parameter (γ) to 5.19, the mean (μ) and standard deviation (σ) of the normal consumption growth shocks to 2.3% and 1%, the jump intensity (ϖ) to 1.7%, the mean jump size (θ) to -38%, and the jump size volatility (δ) to 25%. The figure reports the exchange rate volatility (defined as $\sqrt{2L}$, where L denotes the average entropy) and the currency risk premium in levels from the perspective of the home investor, $E_t\left[rx_{t+1}^{FX}\right] + L_t\left[rx_{t+1}^{FX}\right]$, for the admissible combinations of the jump parameters $\delta_d \leq 2\delta$. The gray area represents the empirical counterpart of each moment.

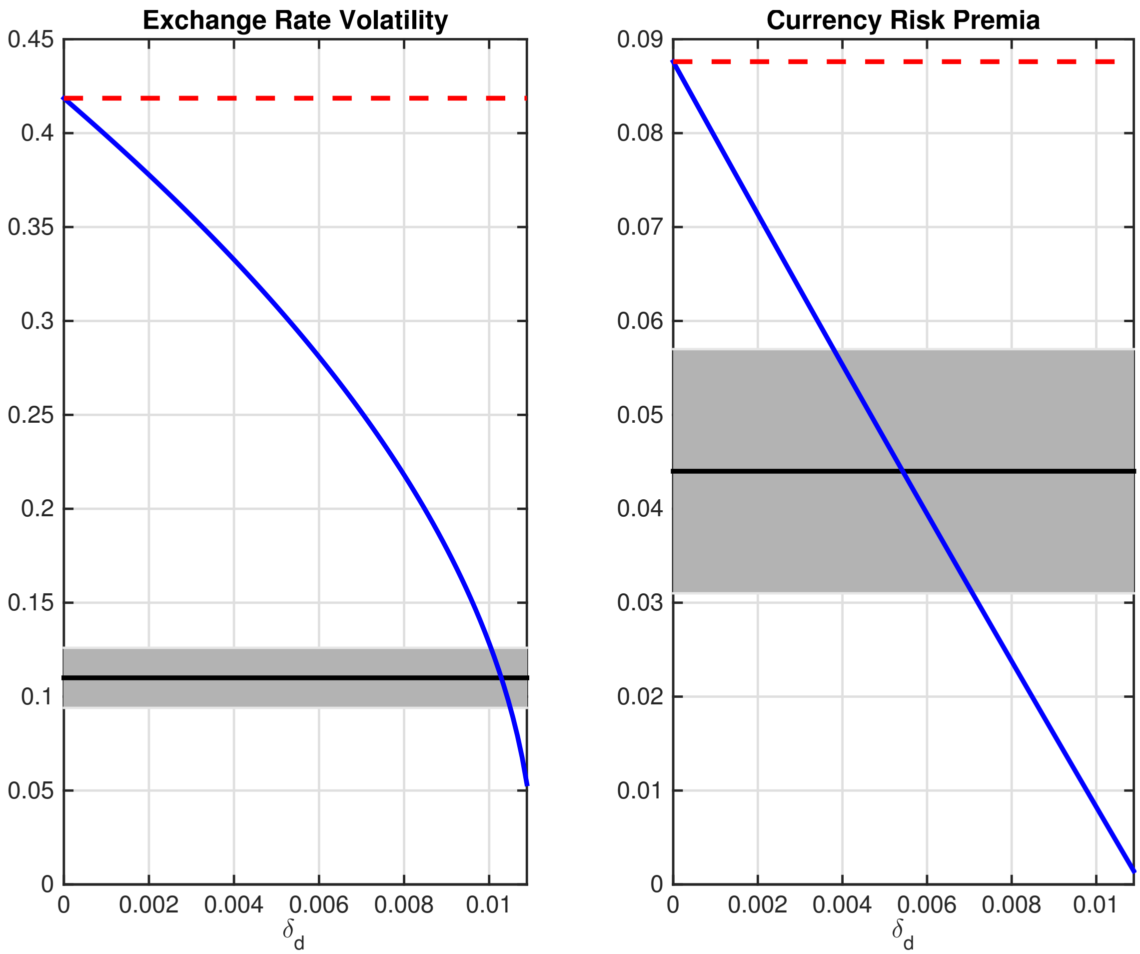

Figure 5: Exchange Rate Volatility and Risk Premia in an option-based Merton (1976) Model: In the model, the incomplete spanning wedge follows the same Poisson mixture as the jump components of consumption growth. Each jump triggers a draw from a normal distribution with mean θ and variance δ^2 for the domestic agent, with mean θ^* and variance δ^{*2} for the foreign agent, and with mean θ_d and variance δ_d^2 for the wedge. We assume that the wedge does not introduce non-stationarity in exchange rates ($\theta_d = 0$), that the two countries' consumption growth processes are symmetric ($\theta = \theta^*$, $\delta = \delta^*$) and that the jump components are not correlated across countries. We follow Backus, Chernov, and Martin (2011) and set the risk aversion parameter (γ) to 8.70, the mean (μ) and standard deviation (σ) of the normal consumption growth shocks to 3.03% and 2.53%, the jump intensity (ϖ) to 139%, the mean jump size (θ) to -0.74%, and the jump size volatility (δ) to 1.91%. The figure reports the exchange rate volatility (defined as $\sqrt{2L}$, where L denotes the average entropy) and the currency risk premium in levels from the perspective of the home investor, $E_t\left[rx_{t+1}^{FX}\right] + L_t\left[rx_{t+1}^{FX}\right]$, for the admissible combinations of the jump parameters $\delta_d \leq 2\delta$. The gray area represents the empirical counterpart of each moment.

own factor z_t^* and shocks u_{t+1}^*.

$$
\begin{aligned}
-\log M_{t+1}^* &= \alpha + \chi z_t^* + \sqrt{\gamma^* z_t^*} u_{t+1}^*, \\
z_{t+1}^* &= (1-\phi)\theta + \phi z_t^* - \sigma \sqrt{z_t} u_{t+1}^*,
\end{aligned}
$$

As noted in Equation (12), since the SDF is lognormal, the maximum Sharpe ratios at home and abroad are $std_t(m_{t+1}) = \sqrt{\gamma z_t}$, and $std_t(m_{t+1}^*) = \sqrt{\gamma^* z_t^*}$, respectively. The real version of this CIR model with $\chi = 0$ is isomorphic to a model in which the domestic (foreign) representative agent has power utility preferences over consumption with CRRA coefficient $\sqrt{\gamma}$ ($\sqrt{\gamma^*}$) and aggregate consumption growth is heteroskedastic.

We assume that domestic investors can trade at least one risky domestic asset (e.g., a longer maturity bond) and the one-period risk-free bond, but they can only trade the foreign risk-free bond. They cannot trade any foreign risky assets. All domestic shocks are spanned, but not the foreign shocks.

When markets are complete, the volatility of exchange rate changes is simply equal to $var_t(\Delta s_{t+1}) = \gamma z_t + \gamma^* z_t^*$. In order to describe the class of potential wedges, we define the target volatility of the incomplete spanning exchange rate as $var_t(\Delta s_{t+1}) = \kappa z_t + \kappa^* z_t^*$. As noted in Corollary 1, the implied volatility of the incomplete spanning exchange rate process is then equal to $var_t(\Delta s_{t+1}) = \gamma z_t + \gamma^* z_t^* - var_t(\eta_{t+1})$, which implies that the volatility of the wedge is $var_t(\eta_{t+1}) = (\gamma - \kappa) z_t + (\gamma^* - \kappa^*) z_t^*$. The following result defines the incomplete markets wedge that matches the desired volatility of the exchange rates while satisfying all the restrictions of Proposition 1.

Result 5. *In the CIR model with country-specific factors that define the domestic m_{t+1} and foreign m_{t+1}^* log stochastic discount factors, incomplete spanning leads to a wedge η_{t+1} and an exchange rate process S_t that satisfies $\Delta s_{t+1} = \eta_{t+1} + m_{t+1}^* - m_{t+1}$ with variance $var_t(\Delta s_{t+1}) =$*

$\kappa z_t + \kappa^* z_t^*$, *where* η_t *follows:*

$$\begin{aligned}\eta_{t+1} = \psi z_t + \psi^* z_t^* \quad &- \quad \sqrt{(\gamma - \lambda) z_t} u_{t+1} + \sqrt{(\gamma^* - \lambda^*) z_t^*} u_{t+1}^* \\ &+ \quad \sqrt{(\lambda - \kappa) z_t} \epsilon_{t+1} + \sqrt{(\lambda^* - \kappa^*) z_t^*} \epsilon_{t+1}^*. \end{aligned} \tag{26}$$

where $\epsilon_{t+1} \sim \mathbb{N}(0,1)$ *and* $\epsilon_{t+1}^* \sim \mathbb{N}(0,1)$ *are i.i.d., and where the parameters* λ, λ^*, ψ, *and* ψ^* *satisfy* $\kappa \leq \lambda \leq \gamma$, $\kappa^* \leq \lambda^* \leq \gamma^*$, *as well as:*

$$\kappa = \gamma - \sqrt{\gamma}\sqrt{\gamma - \lambda}, \tag{27}$$

$$\kappa^* = \gamma^* - \sqrt{\gamma^*}\sqrt{\gamma^* - \lambda^*} \tag{28}$$

$$\psi = -\frac{1}{2}(\gamma - \kappa), \tag{29}$$

$$\psi^* = \frac{1}{2}(\gamma^* - \kappa^*). \tag{30}$$

The class of incomplete spanning models built on the CIR framework has only two degrees of freedom, described by the two parameters κ and κ^*. Again, these two parameters determine the exchange rate volatility. Once they are chosen, the law of motion of the incomplete spanning wedge is entirely determined. Equations (27) and (28) implicitly pin down the parameters λ and λ^*. As Equations (29) and (30) show, the drift term in the η process is not a free parameter either, it is determined by the other parameters of the model.[9]

The incomplete markets wedges leave the domestic and foreign term structure unchanged. The term $(\gamma - \lambda)$ measures the exchange rate's exposure to spanned shocks, while $(\lambda - \kappa)$ measures the exposure to unspanned shocks. If we allow the domestic investor to trade any foreign risky bond, then the wedges are zero again: $\kappa = \gamma = \lambda$ and $\kappa^* = \gamma^* = \lambda^*$, because we need to impose two additional orthogonality conditions given by Equation (13) between log returns and η. This result is intuitive: if there are as many assets as exogenous shocks, markets are complete.

[9] In the symmetric case, where $\gamma = \gamma^*$ (the two SDFs react in the same proportion to exogenous shocks) and $\kappa = \kappa^*$ (the exchange rate volatility exhibit the same sensitivity to the two state variables), then Equations (29) and (30) imply that $\psi = -\psi^\star$, and the drift term is zero on average ($E[\mu_{t,\eta}] = 0$). In the symmetric case, on average, the wedge has no impact on exchange rates.

The key result is that the drift of the wedge is determined by the rest of the model; it is no longer a free parameter. Once we impose these dynamic no-arbitrage restrictions on the drift term, the effect on currency risk premia is unambiguous. When markets are complete, the log currency risk premium is given by $E_t[rx_{t+1}^{FX}] = \frac{1}{2}(\gamma z_t - \gamma^* z_t^*)$, while the currency risk premium in levels is given by γz_t. When markets are incomplete, the risk premium in levels is always smaller than in complete markets because $\kappa \leq \gamma$.

Result 6. *In the incomplete market model described in Result 5, the risk premium in logs on a long position in foreign currency is:*

$$E_t[rx_{t+1}^{FX}] = \frac{1}{2}\left[\kappa z_t - \kappa^* z_t^*\right].$$

The risk premium in levels on a long position in foreign currency is always smaller in incomplete markets than in complete markets:

$$E_t[rx_{t+1}^{FX}] + \frac{1}{2} var_t[rx_{t+1}^{FX}] = \kappa z_t \leq \gamma z_t.$$

In the incomplete market model, the Fama slope coefficient in a regression of exchange rates ($-\Delta s_{t+1}$) on the interest rate difference $r_t^* - r_t$ is:

$$\frac{cov(-\Delta s_{t+1}, r_t^* - r_t)}{var(r_t^* - r_t)} = 1 + \frac{1}{2}\frac{\kappa(\chi - \frac{1}{2}\gamma) + \kappa^*(\chi - \frac{1}{2}\gamma^*)}{(\chi - \frac{1}{2}\gamma)^2 + (\chi - \frac{1}{2}\gamma^*)^2}.$$

If incomplete spanning reduces the standard deviation of exchange rates by 50% ($\sqrt{\kappa/\gamma} = 0.5$), then the currency risk premium is reduced by a factor of 0.25 ($\kappa/\gamma = 0.25$, implying a reduction by 75%).[10] Since a real version of the CIR model is isomorphic to the Consumption-CAPM with heteroscedastic consumption growth, this result implies that incomplete spanning

[10]Currency Sharpe ratios decrease as well, since for all z_t, z_t^*:

$$\frac{E_t[rx_{t+1}^{FX}] + \frac{1}{2} var_t[rx_{t+1}^{FX}]}{std_t(\Delta s_{t+1})} = \frac{\kappa z_t}{\sqrt{\kappa z_t + \kappa z_t^*}} = \sqrt{\kappa}\frac{z_t}{\sqrt{z_t + z_t^*}} \leq \sqrt{\gamma}\frac{z_t}{\sqrt{z_t + z_t^*}}.$$

effectively reduces the representative agent's risk aversion coefficient when pricing currency risk, but not for other risk sources.

To quantitatively illustrate the trade-off between exchange rate volatility and risk premia, we adopt the following parameters for the two countries: $\lambda_d = -1.07$, $\gamma = \lambda_d^2$, $\theta = 0.004428$, $\phi = 0.976$, $\alpha = 0$, $\chi = -1 + \lambda_d^2/2$, $\sigma = 0.008356$. These parameters match the mean short-term interest rate rate, its volatility, and its autocorrelation. They are close to those used in Backus, Foresi, and Telmer (1998): the only difference is that we defined $\chi = -1 + \lambda_d^2/2$ (instead of $\chi = 1 + \lambda_d^2/2$) in order to obtain counter-cyclical short-term interest rates, a necessary feature to replicate the uncovered interest rate (UIP) puzzle in this class of models.

Figure 6 reports the annualized volatility of the exchange rate and the UIP slope coefficients for all admissible combinations of the parameters κ. The first panel plots the parameters κ and λ against the annualized volatility of the wedge, $std_t(\eta_{t+1})$. The second panel plots the annualized volatility of the exchange rate. The third panel plots the UIP slope coefficient in a regression of exchange rates $(-\Delta s_{t+1})$ on the interest rate difference $r_t^* - r_t$. UIP implies that this slope coefficient is one; in the data, it is statistically different from one and often negative. As the volatility of the wedge increases, the exchange rate volatility decreases. It can reach its empirical value, but only at the cost of driving the UIP slope coefficients to one. The fourth panel reports the currency risk premium. In the model, as shown in Corollary 6, it varies with the state variable z. In order to focus on potentially large values, it is here evaluated at the mean plus two standard deviations of the state variable z. Even in this very favorable case, when the exchange rate volatility reaches its empirical value, the currency risk premium is zero.

These conclusions do not depend on the country-specific nature of the factors. When we include common factors, we find that incomplete spanning still lowers the currency risk premium in levels and also forces the UIP regression coefficient to one. We analyze this general case with common factors in the Appendix, along with a version of the Lucas (1982) model with hetereroskedastic consumption growth. In this model, we show that incompleteness lowers the risk aversion of the representative agent by the same percentage amount as it lowers exchange rate volatility.

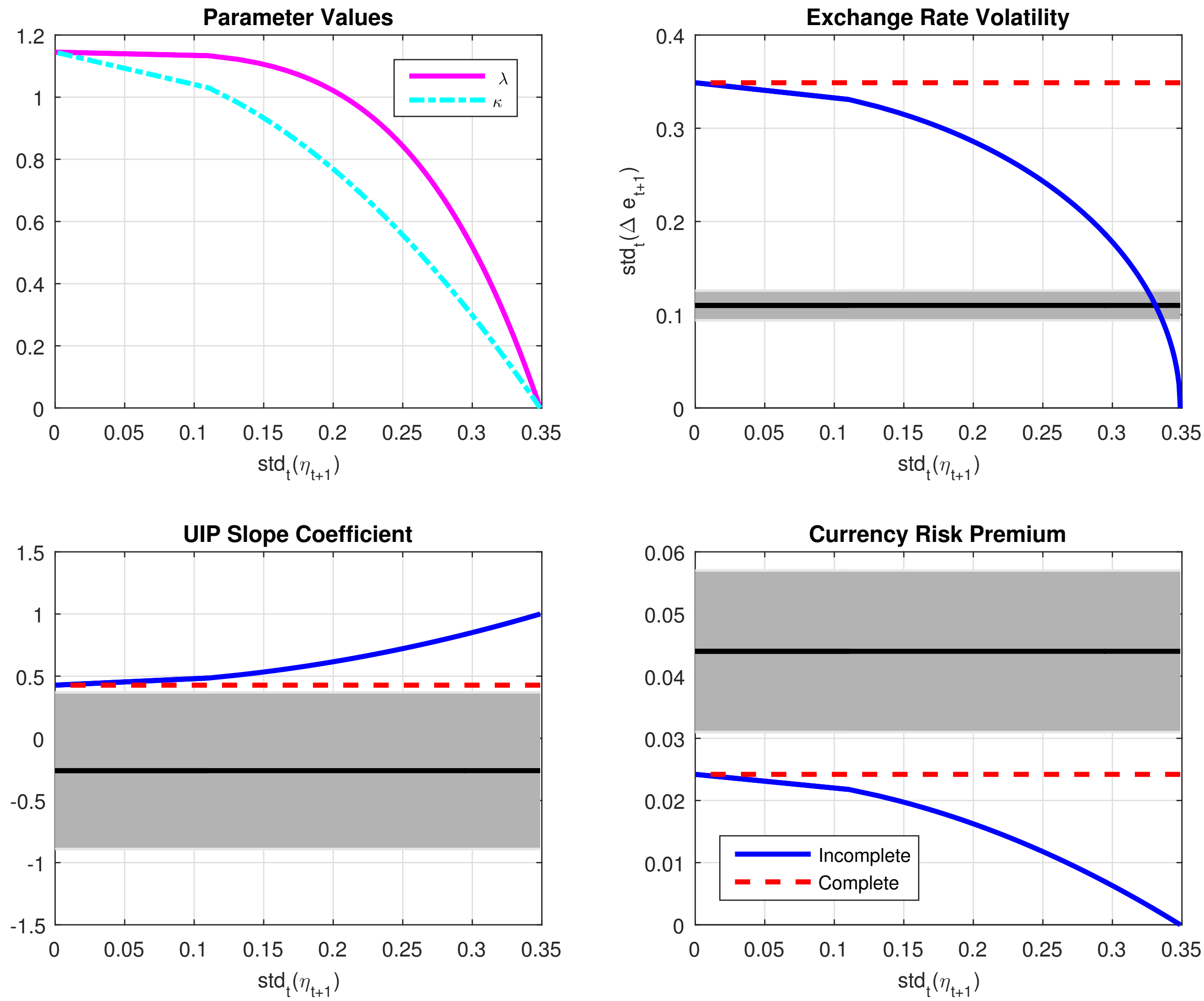

Figure 6: Cox, Ingersoll, and Ross (1985) Example: The figure reports the wedge parameters, the annualized volatility of the exchange rate, the UIP slope coefficient, and the currency risk premium for different volatilities of the incomplete spanning wedge. The first panel plots the parameters κ and λ that characterize the wedge against its annualized volatility, $std_t(\eta_{t+1})$. The second panel plots the annualized volatility of the exchange rate. The third panel plots the UIP slope coefficient in a regression of exchange rates on the foreign minus domestic interest rates. If the uncovered interest rate parity were to hold, the UIP slope coefficient would be one. The fourth panel reports the currency risk premium evaluated at the mean plus two standard deviations of the state variable. We adopt the following parameters for the two countries: $\lambda_d = -1.07$, $\gamma = \lambda_d^2$, $\theta = 0.004428$, $\phi = 0.976$, $\alpha = 0$, $\chi = -1 + \lambda_d^2/2$, $\sigma = 0.008356$.

4.4 Existing Incomplete Market Models

We end this paper with an overview of the exchange rate literature on incomplete markets.

Pavlova and Rigobon (2012) We start of our review of the existing incomplete market models with the work of Pavlova and Rigobon (2012) because it satisfies our Assumptions 1 and 2 and delivers closed-form expressions. In their model, agents trade three assets (domestic and foreign stocks and an international bond), but are subject to four shocks (two endowment shocks and two preference shocks) and markets are thus incomplete. Log preferences lead to the following equilibrium real exchange rate, Q_t^{IM}:

$$Q_t^{IM} = \frac{\theta_H(t) + \frac{W_F(t)}{W_H(t)}(1-\theta_F)}{1-\theta_H(t) + \frac{W_F(t)}{W_H(t)}\theta_F}\frac{Y_t^*}{Y_t},$$

where Y_t and Y_t^* are the domestic and foreign endowments, $\theta_H(t)$ denotes the domestic intra-temporal preference shocks, and $W_H(t)$ and $W_F(t)$ denote domestic and foreign wealth. In this model, the wealth ratio $W_F(t)/W_H(t)$ is time-varying. In an earlier contribution, Pavlova and Rigobon (2007) consider a similar model, but with only one preference shocks, and thus as many shocks as assets. In Pavlova and Rigobon (2007), where markets are complete, the wealth ratio is constant and the same expression holds for the equilibrium real exchange rate, Q_t^{CM}. The two models are different: market incompleteness clearly changes the consumption allocation. But the incomplete market model can be thought as introducing an additional source of variation in exchange rates. The ratio of the incomplete to complete market value of exchange rate is:

$$\frac{Q_t^{IM}}{Q_t^{CM}} = \frac{\theta_H(t) + \frac{W_F(t)}{W_H(t)}(1-\theta_F)}{1-\theta_H(t) + \frac{W_F(t)}{W_H(t)}\theta_F} \times \frac{1-\theta_H(t) + \frac{W_F}{W_H}\theta_F}{\theta_H(t) + \frac{W_F}{W_H}(1-\theta_F)},$$

The FX wedge η_{t+1} introduced in Equation (7) captures the percentage change in this ratio ($\eta_{t+1} = \Delta q_t^{IM} - \Delta q_t^{CM}$). Our approach offers a way to study the effects of incomplete market models on exchange rate puzzles without committing to any preferences or frictions. In Pavlova and Rigon (2012), log preferences lead to smooth SDFs and thus low Sharpe ratio, contrary to the data and our Assumption 3.

Chari et al. (2002), Corsetti et al. (2008), Gabaix and Maggiori (2015) We turn now to three interesting incomplete market models and show how they differ from our assumptions.

We simulate the incomplete market model of Chari, Kehoe, and McGrattan (2002), thanks to the code provided by the authors. In their model, the utility for period t is

$$U(c,l,M/P) = \frac{1}{1-\sigma}\left[\left(\omega c^{\frac{\eta-1}{\eta}} + (1-\omega)\left(\frac{M}{P}\right)^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}\right]^{1-\sigma} + \psi\frac{(1-l)^{1-\gamma}}{1-\gamma}$$

where c denotes consumption, l labor, M/P real money balances, and σ, ω, η, γ, and ψ are model parameters (η here denotes the interest rate elasticity). The marginal utility of consumption is thus

$$U_c(c,l,M/P) = \omega c^{-\frac{1}{\eta}}\left[\left(\omega c^{\frac{\eta-1}{\eta}} + (1-\omega)\left(\frac{M}{P}\right)^{\frac{\eta-1}{\eta}}\right)^{\frac{1}{\eta-1}}\right]^{-\sigma}$$

and the SDF is $M_{t+1} = \beta U_c(c_{t+1}, M_{t+1}/P_{t+1})/U_c(c_t, M_t/P_t)$. We slightly modify the code to compute the SDF, and use the following parameter values: $\beta = 0.99$, $\eta = 0.39$, $\omega = 0.94$, and $\sigma = 5$ (as reported in their Table 5). The different versions of the incomplete market model (with or without multiple shocks, with one-period or more sticky prices) deliver standard deviations of log SDFs that range from 0.14 to 0.17. The model thus does not deliver Sharpe ratios in line with our Assumption 3.

The same happens in the incomplete market model of Corsetti, Dedola, and Leduc (2008): in their model, the quarterly volatility of the stochastic discount factor is 0.75% (and 0.52% when HP-filtered as for the other variables in the paper).[11] The model thus implies an annualized maximal Sharpe ratio of 0.015 that is an order of magnitude smaller than in the data (0.5). While the model offers many interesting insights, it is not built to describe risk premia.

The incomplete market model of Gabaix and Maggiori (2015) does not satisfy Assumption 2: in their model, domestic agents cannot buy foreign risk-free bonds directly, they have to transact through a global intermediary.

[11] We thank Sylvain Leduc for explaining this to us.

Bakshi, Cerrato, and Crosby (2017) Building on our work, Bakshi, Cerrato, and Crosby (2017) study the ability of additive wedges to address exchange rate facts. Is this different from our multiplicative wedge? Only in the pathological case of negative stochastic discount factors. To see this, start from their definition of the wedge:

$$\frac{S_{t+1}}{S_t} = \frac{M^*_{t+1}}{M_{t+1}} + \lambda_{t+1}, \tag{31}$$

which is equivalent to:

$$\frac{S_{t+1}}{S_t} = \frac{M^*_{t+1}}{M_{t+1}}(1 + \lambda_{t+1}\frac{M_{t+1}}{M^*_{t+1}}). \tag{32}$$

By comparison, our multiplicative wedge is defined as:

$$\frac{S_{t+1}}{S_t} = \frac{M^*_{t+1}}{M_{t+1}}e^{\eta_{t+1}}. \tag{33}$$

Recall that our results are valid for any pair of stochastic discount factors. By introducing a multiplicative wedge, we thus only consider the cases where the term $1 + \lambda_{t+1}\frac{M_{t+1}}{M^*_{t+1}}$ is positive. But can it be negative? Not if $\frac{M^*_{t+1}}{M_{t+1}}$ is positive: since exchange rates are positive numbers, Equation (32) implies that $1 + \lambda_{t+1}\frac{M_{t+1}}{M^*_{t+1}}$ is positive. If $\frac{M^*_{t+1}}{M_{t+1}}$ is negative, then there are some additive wedges λ_{t+1} that are not described by our multiplicative wedge $e^{\eta_{t+1}}$. But this case is pathological: in a utility-based framework, it assumes that agents' utility would decrease when consumption increases. A long tradition in economics only consider models with positive marginal utilities, i.e. utility increases when consumption increases.

5 Conclusion

Our paper investigates whether incomplete spanning in international financial markets can account for the behavior of exchange rates. To answer this question, we allow for a great deal of market incompleteness by only enforcing the Euler equations on domestic and foreign risk-free rates. To help resolve the currency volatility puzzle, the quantity of unspanned risk needed in currency markets is of the same size as the maximum Sharpe ratio. To increase the currency risk premium requires the introduction of a predictable exchange rate component. In a lognormal world, incomplete spanning does not address the exchange rate cyclicality: the covariance between exchange rate changes and relative SDFs remains positive, and the corresponding slope coefficient remains at its complete market value of one.

The limits of incomplete spanning underlines the robustness of the key exchange rate puzzles. In the future, the solutions to these puzzles may involve two ingredients. As suggested by complete market models, stochastic discount factors may be very highly correlated, even if macroeconomic series are not. To support this view further, researchers need to find direct evidence of such high correlations. In the realm of segmented markets, models that segment international currency markets by only allowing a subset of investors (see, e.g., Chien, Lustig, and Naknoi, 2015 and Dou and Verdelhan, 2015) to trade a complete (or incomplete) menu of international securities are promising. These models sever the link between aggregate quantities and real exchange rates by concentrating aggregate risk among a small pool of investors. But these segmented markets models face a challenging measurement test: researchers need to show that changes in exchange rates are highly correlated with the marginal utility growth of these market participants.

References

Alvarez, F., A. Atkeson, and P. J. Kehoe (2002): "Money, Interest Rates and Exchange Rates with Endogenously Segmented Markets," *Journal of Political Economy*, 110(1), 73–112.

——— (2009): "Time-Varying Risk, Interest Rates and Exchange Rates in General Equilibrium," *Review of Economic Studies*, 76, 851 – 878.

Alvarez, F., and U. Jermann (2005): "Using Asset Prices to Measure the Measure the Persistence of the Marginal Utility of Wealth," *Econometrica*, 73(6), 1977–2016.

Amador, M., J. Bianchi, L. Bocola, and F. Perri (2017): "Exchange rate policies at the zero lower bound," Discussion paper, National Bureau of Economic Research.

Bacchetta, P., and E. van Wincoop (2006): "Can Information Heterogeneity Explain the Exchange Rate Determination Puzzle?," *American Economic Review*, 96(3), 552–576.

Backus, D., S. Foresi, and C. Telmer (1998): "Discrete-Time Models of Bond Pricing," Discussion paper, National Bureau of Economic Research.

——— (2001): "Affine Term Structure Models and the Forward Premium Anomaly," *Journal of Finance*, 56(1), 279–304.

Backus, D., and G. Smith (1993): "Consumption and Real Exchange Rates in Dynamic Economies with Non-Traded Goods," *Journal of International Economics*, 35, 297–316.

Backus, D. K., M. Chernov, and N. Boyarchenko (2016): "Term structures of asset prices and returns," *Journal of Financial Economics, forthcoming.*

Backus, D. K., M. Chernov, and I. Martin (2011): "Disasters Implied by Equity Index Options," *Journal of Finance*, 66, 1967–2009.

Bansal, R. (1997): "An Exploration of the Forward Premium Puzzle in Currency Markets," *Review of Financial Studies*, 10, 369–403.

Bansal, R., and I. Shaliastovich (2012): "A Long-Run Risks Explanation of Predictability Puzzles in Bond and Currency Markets," *Review of Financial Studies*.

Bates, D. S., et al. (1988): *Pricing options under jump-diffusion processes*. Citeseer.

Bekaert, G. (1996): "The Time Variation of Risk and Return in Foreign Exchange Markets: A General Equilibrium Perspective," *The Review of Financial Studies*, 9(2), 427–470.

Brandt, M. W., J. Cochrane, and P. Santa-Clara (2006): "International Risk-Sharing is Better Than You Think, or Exchange Rates are Too Smooth," *Journal of Monetary Economics*, 53(4), 671–698.

Brennan, M. J., and Y. Xia (2006): "International Capital Markets and Foreign Exchange Risk," *Review of Financial Studies*, 19(3), 753–795.

Bruno, V., and H. S. Shin (2014): "Cross-Border Banking and Global Liquidity," *The Review of Economic Studies*, p. rdu042.

Chari, V. V., P. J. Kehoe, and E. R. McGrattan (2002): "Can Sticky Price Models Generate Volatile and Persistent Real Exchange Rates?," *The Review of Economic Studies*, 69(240), 533–564.

Chari, V. V., P. J. Kehoe, and E. R. McGrattan (2007): "Business cycle accounting," *Econometrica*, 75(3), 781–836.

Chien, Y., H. Lustig, and K. Naknoi (2015): "Why Are Exchange Rates So Smooth? A Segmented Asset Markets Explanation," Working Paper UCLA Anderson School of Management.

Cochrane, J. H. (2005): *Asset Pricing*. Princeton University Press, Princeton, N.J.

Colacito, R., M. Croce, S. Ho, and P. Howard (2014): "BKK the EZ way," *Working Paper, University of North Carolina.*

Colacito, R., and M. M. Croce (2011): "Risks For The Long Run And The Real Exchange Rate," *Journal of Political Economy*, 119(1), 153–182.

——— (2013): "International Asset Pricing with Recursive Preferences," *The Journal of Finance*, 68(6), 2651–2686.

Colacito, R., M. M. Croce, F. Gavazzoni, and R. C. Ready (2017): "Currency risk factors in a recursive multi-country economy," *The Journal of Finance, forth.*

Corsetti, G., L. Dedola, and S. Leduc (2008): "International Risk Sharing and the Transmission of Productivity Shocks," *Review of Economic Studies*, 75(2), 443 – 473.

Cox, J., J. Ingersoll, and S. Ross (1985): "An Intertemporal General Equilibrium Model of Asset Pricing," *Econometrica*, 53, 363–384.

Croce, M. M., T. T. Nguyen, and L. Schmid (2013): "Global entropy," *Working Paper, University of North Carolina.*

Dou, W. W., and A. Verdelhan (2015): "The Volatility of International Capital Flows and Foreign Assets," Working Paper MIT Sloan.

Du, W., A. Tepper, and A. Verdelhan (2017): "Deviations from Covered Interest Rate Parity," *Journal of Finance, forth.*

Fama, E. F. (1984): "Forward and Spot Exchange Rates," *Journal of Monetary Economics*, 14, 319–338.

Farhi, E., and X. Gabaix (2015): "Rare Disasters and Exchange Rates," *The Quarterly Journal of Economics, forthcoming.*

Farhi, E., and I. Werning (2017): "Fiscal unions," *American Economic Review*, 107(12), 3788–3834.

Favilukis, J., L. Garlappi, and S. Neamati (2015): "The Carry Trade and Uncovered Interest Parity when Markets are Incomplete," Working Paper UBC.

Frachot, A. (1996): "A Reexamination of the Uncovered Interest Rate Parity Hypothesis," *Journal of International Money and Finance*, 15(3), 419–437.

Gabaix, X., and M. Maggiori (2015): "International Liquidity and Exchange Rate Dynamics," *The Quarterly Journal of Economics*, 130(3), 1369–1420.

Gavazzoni, F., and A. M. Santacreu (2015): "International r&d spillovers and asset prices," *INSEAD–Economics and Political Science.*

Graveline, J. J., and S. Joslin (2011): "G10 Swap and Exchange Rates," Working Paper University of Southern California.

Hansen, L. P., and R. J. Hodrick (1980): "Forward Exchange Rates as Optimal Predictors of Future Spot Rates: An Econometric Analysis," *Journal of Political Economy*, 88(5), 829–853.

Hansen, L. P., and R. Jagannathan (1991): "Implications of Security Market Data for Models of Dynamic Economies," *Journal of Political Economy*, 99(2), 225–262.

Hodrick, R., and M. Vassalou (2002): "Do We Need Multi-Country Models to Explain Exchange Rate and Interest Rate and Bond Return Dynamics?," *Journal of Economic Dynamics and Control*, 26(7), 1275–1299.

Itskhoki, O., and D. Mukhin (2017): "Exchange rate disconnect in general equilibrium," Discussion paper, National Bureau of Economic Research.

Kollmann, R. M. (1991): "Essays on International Business Cycles," Ph.D. thesis, University of Chicago, Department of Economics.

Lewis, K. K. (1995): "Puzzles in International Financial Markets," in *Handbook of International Economics*, ed. by G. Grossman, and K. Rogoff, pp. 1913–1971. Elsevier Science B.V., Amsterdam.

Lucas, R. E. J. (1982): "Interest rates and currency prices in a two country world," *Journal of Monetary Economics*, 10, 335–360.

Lustig, H., N. Roussanov, and A. Verdelhan (2011): "Common Risk Factors in Currency Markets," *Review of Financial Studies*, 24 (11), 3731–3777.

——— (2014): "Countercyclical Currency Risk Premia," *Journal of Financial Economics*, 111 (3), 527–553.

Lustig, H., A. Stathopoulos, and A. Verdelhan (2017): "The Term Structure of Currency Carry Trade Risk Premia," *Working Paper*.

Lustig, H., and A. Verdelhan (2007): "The Cross-Section of Foreign Currency Risk Premia and Consumption Growth Risk," *American Economic Review*, 97(1), 89–117.

Maggiori, M. (2017): "Financial Intermediation, International Risk Sharing, and Reserve Currencies," *American Economic Review*, 107(10), 3038–71.

Martin, I. W. (2013): "Consumption-based asset pricing with higher cumulants," *The Review of Economic Studies*, 80(2), 745–773.

Maurer, T. A., and N.-K. Tran (2016): "Entangled Risks in Incomplete FX Markets," Working Paper.

Merton, R. C. (1976): "Option pricing when underlying stock returns are discontinuous," *Journal of financial economics*, 3(1), 125–144.

Naik, V., and M. Lee (1990): "General equilibrium pricing of options on the market portfolio with discontinuous returns," *Review of Financial Studies*, 3(4), 493–521.

Nakamura, E., J. Steinsson, R. J. Barro, and J. F. Ursua (2013): "Crises and Recoveries in an Empirical Model of Consumption Disasters," *American Economic Journal: Macroeconomics*, 5(3), 35–74.

Pavlova, A., and R. Rigobon (2007): "Asset Prices and Exchange Rates," *Review of Financial Studies*, 20 (4), 1139–1180.

——— (2010): "An Asset-Pricing View of External Adjustment," *Journal of International Economics*, 80(1), 144–156.

——— (2012): "Equilibrium Portfolios and External Adjustment Under Incomplete Markets," AFA 2009 San Francisco Meetings Paper.

Ross, S. A. (1978): "A Simple Approach to the Valuation of Risky Streams," *Journal of Business*, pp. 453–475.

Sarno, L., P. Schneider, and C. Wagner (2012): "Properties of Foreign Exchange Risk Premiums," *Journal of Financial Economics*, 105(2), 279 – 310.

Schmitt-Grohé, S., and M. Uribe (2016): "Downward nominal wage rigidity, currency pegs, and involuntary unemployment," *Journal of Political Economy*, 124(5), 1466–1514.

Stathopoulos, A. (2017): "Asset Prices and Risk Sharing in Open Economies," *The Review of Financial Studies*, 30(2), 363–415.

Tryon, R. (1979): "Testing for Rational Expectations in Foreign Exchange Markets," International Finance Discussion Papers.

Appendix

For Online Publication

Section A presents the proof of our main general results. Section B reports the quantitative implications of our results when the home and foreign log SDFs exhibit different volatilities. Section C studies three examples: a simple consumption-based example; a Cox, Ingersoll, and Ross (1985) model with common factors; a consumption-based example with heteroscedasticity. Section D reports summary statistics on the exchange rate entropy.

A Proofs of Main Results

In this section, we gather all the proofs of the main results in the text, in the order they appear there. We distinguish between the propositions and their corollaries, which are model-free findings, and the results, which are model-specific.

Proof of Proposition 1:

Proof. We start from the domestic investor's Euler equation for the foreign risk-free asset, and the foreign investor's Euler equation for the domestic risk-free asset respectively:

$$\begin{aligned} E_t\left(\widehat{M}^*_{t+1}\right) &= E_t\left(M_{t+1}\frac{S_{t+1}}{S_t}\right) = E_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) = 1/R^{f,*}_t, \\ E_t\left(M_{t+1}\right) &= E_t\left(M^*_{t+1}\frac{S_t}{S_{t+1}}\right) = E_t\left(M_{t+1}\exp(-\eta_{t+1})\right) = 1/R^{f,}_t. \end{aligned}$$

By using conditional joint log normality of the foreign SDF and $exp(\eta)$, the first Euler equation implies that:

$$\begin{aligned} E_t\left(\log M^*_{t+1}\right) + \frac{1}{2}Var_t\left(\log M^*_{t+1}\right) &= E_t\left(\log M^*_{t+1}\right) + \mu_{t,\eta} + \frac{1}{2}Var_t\left(\log M^*_{t+1}\right) \\ &+ \frac{1}{2}Var_t\left(\eta_{t+1}\right) + covar_t(\eta_{t+1}, \log M^*_{t+1}), \end{aligned}$$

where $\mu_{t,\eta} = E_t\left(\eta_{t+1}\right)$. This implies that $covar_t\left(m^*_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} - 0.5var_t\left(\eta_{t+1}\right)$. We move on to the second equation. The second Euler equation for the domestic risk-free asset implies that:

$$\begin{aligned} E_t\left(\log M_{t+1}\right) + \frac{1}{2}Var_t\left(\log M_{t+1}\right) &= E_t\left(\log M_{t+1}\right) - \mu_{t,\eta} + \frac{1}{2}Var_t\left(\log M_{t+1}\right) \\ &+ (1/2)Var_t\left(\eta_{t+1}\right) - covar_t(\eta_{t+1}, \log M_{t+1}). \end{aligned}$$

This implies that $covar_t\left(m_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} + 0.5var_t\left(\eta_{t+1}\right)$.

The inequality restrictions on $\mu_{t,\eta}$ follow directly from the Cauchy-Schwarz inequality for (1) $|covar_t\left(m^*_{t+1}, \eta_{t+1}\right)| \leq std_t\left(m^*_{t+1}\right)std_t\left(\eta_{t+1}\right)$ and (2) $|covar_t\left(m_{t+1}, \eta_{t+1}\right)| \leq std_t\left(m_{t+1}\right)std_t\left(\eta_{t+1}\right)$. Finally, we also impose that (3):

$$|covar_t\left(m^*_{t+1} - m_{t+1}, \eta_{t+1}\right)| \leq std_t\left(m^*_{t+1} - m_{t+1}\right)std_t\left(\eta_{t+1}\right).$$

When $\mu_{t,\eta} \leq -(1/2)var_t\left(\eta_{t+1}\right)$, the first inequality implies that:

$$-(\mu_{t,\eta} + \frac{1}{2}var_t\left(\eta_{t+1}\right)) \leq std_t\left(m^*_{t+1}\right)std_t\left(\eta_{t+1}\right).$$

This in turn implies that:

$$-(\mu_{t,\eta}) \leq std_t\left(m^*_{t+1}\right)std_t\left(\eta_{t+1}\right) + \frac{1}{2}var_t\left(\eta_{t+1}\right)).$$

When $\mu_{t,\eta} \geq -(1/2)var_t\left(\eta_{t+1}\right)$, the first inequality implies that:

$$\mu_{t,\eta} + \frac{1}{2}var_t\left(\eta_{t+1}\right) \leq std_t\left(m^*_{t+1}\right)std_t\left(\eta_{t+1}\right).$$

This in turn implies that:

$$\mu_{t,\eta} \leq std_t\left(m^*_{t+1}\right) std_t\left(\eta_{t+1}\right) - \frac{1}{2} var_t\left(\eta_{t+1}\right).$$

Next, we turn to the second inequality. When $\mu_{t,\eta} \geq (1/2) var_t\left(\eta_{t+1}\right)$, the second inequality implies that:

$$\mu_{t,\eta} - \frac{1}{2} var_t\left(\eta_{t+1}\right) \leq std_t\left(m_{t+1}\right) std_t\left(\eta_{t+1}\right).$$

This in turn implies that:

$$\mu_{t,\eta} \leq std_t\left(m_{t+1}\right) std_t\left(\eta_{t+1}\right) + \frac{1}{2} var_t\left(\eta_{t+1}\right).$$

When $\mu_{t,\eta} \leq (1/2) var_t\left(\eta_{t+1}\right)$, the second inequality implies that:

$$-(\mu_{t,\eta} - \frac{1}{2} var_t\left(\eta_{t+1}\right)) \leq std_t\left(m_{t+1}\right) std_t\left(\eta_{t+1}\right).$$

This in turn implies that:

$$-\mu_{t,\eta} \leq std_t\left(m_{t+1}\right) std_t\left(\eta_{t+1}\right) - \frac{1}{2} var_t\left(\eta_{t+1}\right).$$

Finally, the third inequality implies that:

$$std_t\left(\eta_{t+1}\right) \leq std_t\left(m^*_{t+1} - m_{t+1}\right).$$

□

Proof of Corollary 1:

Proof. We start from the definition of log changes in exchange rates: $var_t(\Delta s_{t+1}) = var_t(\eta_{t+1} + m^*_{t+1} - m_{t+1})$. This can be simplified to:

$$\begin{aligned} var_t(\Delta s_{t+1}) &= var_t(m_{t+1}) + var_t(m^*_{t+1}) + var_t(\eta_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1}) \\ &- 2cov_t(m_{t+1}, \eta_{t+1}) + 2cov_t(\eta_{t+1}, m^*_{t+1}). \end{aligned}$$

Proposition 1 implies that:

$$\begin{aligned} var_t(\Delta s_{t+1}) &= var_t(m_{t+1}) + var_t(m^*_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1}) \\ &- var_t\left(\eta_{t+1}\right) - var_t\left(\eta_{t+1}\right) + var_t(\eta_{t+1}), \end{aligned}$$

which establishes the result. Finally, we prove the volatility results. The volatility of the log pricing kernel in the foreign country is given by

$$var_t\left(m^*_{t+1} + \eta_{t+1}\right) = var_t(m^*_{t+1}) + var_t(\eta_{t+1}) + 2covar_t(m^*_{t+1}, \eta_{t+1}).$$

The result follows directly from the covariance condition. Note that $covar_t\left(m^*_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} - \frac{1}{2} var_t\left(\eta_{t+1}\right)$.

$$var_t\left(m^*_{t+1} + \eta_{t+1}\right) = var_t(m^*_{t+1}) + var_t(\eta_{t+1}) + 2(-\mu_{t,\eta} - \frac{1}{2} var_t\left(\eta_{t+1}\right)).$$

□

Proof of Corollary 2

Proof. The expression for the log risk premium follows because $covar_t\left(m^*_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} - var_t\left(\eta_{t+1}\right)/2$. The expression for the risk premium in level follows because $var_t[rx^{FX}_{t+1}]/2 = var_t(\Delta s_{t+1})/2$ which is given by:

$$\frac{1}{2} var_t(m_{t+1}) + \frac{1}{2} var_t(m^*_{t+1}) - cov_t(m_{t+1}, m^*_{t+1}) - \frac{1}{2} var_t(\eta_{t+1}).$$

The log risk premium is increased by $\mu_{t,\eta}$ relative to the complete markets case. The foreign investor's log risk premium on domestic currency is naturally the opposite of the one above. The symmetry does not hold in levels because of the usual Jensen term. The foreign investor's risk premium in levels on a long position in domestic currency is given by:

$$E_t[rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] = cov_t(m^*_{t+1}, \Delta s_{t+1}) = var_t\left(m^*_{t+1}\right) - covar_t\left(m^*_{t+1}, m_{t+1}\right) - \frac{1}{2}var_t\left(\eta_{t+1}\right) - \mu_{t,\eta}.$$

□

Proof of Corollary 3

Proof. This result follows immediately from Proposition 1. We subtract the second $covar_t\left(m_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} + 0.5var_t\left(\eta_{t+1}\right)$ from the first covariance condition $covar_t\left(m_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} + 0.5var_t\left(\eta_{t+1}\right)$. That delivers the results. □

Proof of Proposition 2:

Proof. By definition, the conditional entropy of a random variable X_{t+1} is equal to:

$$L_t(X_{t+1}) = \log E_t(X_{t+1}) - E_t(\log X_{t+1})$$

We assume here that both investors have access to risk-free rates. Let us start again from the Euler equation of the foreign investor:

$$\frac{1}{R^{f,*}_t} = E_t\left(M^*_{t+1}\exp(\eta_{t+1})\right)$$

Taking logs leads to:

$$-r^{f,*}_t = \log E\left(M^*_{t+1}\exp(\eta_{t+1})\right) = L_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) + E_t\left(\log M^*_{t+1}\right) + E_t(\eta_{t+1}).$$

But the risk-free rate also satisfies the Euler equation $E\left(M^*_{t+1}R^{f,*}_t\right) = 1$. Taking logs again leads to:

$$\log E\left(M^*_{t+1}R^{f,*}_t\right) = L\left(M^*_{t+1}R^{f,*}_t\right) + E_t\left(\log M^*_{t+1}\right) + r^{f,*}_t = 0$$

Plugging the implied value of the log risk-free rate in the first equation above delivers the result, noting that $L_t(a_t X_{t+1}) = L_t(X_{t+1})$ for any variable a_t known at date t:

$$L\left(M^*_{t+1}\right) + E_t\left(\log M^*_{t+1}\right) = L_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) + E_t\left(\log M^*_{t+1}\right) + E_t(\eta_{t+1}),$$

which simplifies to:

$$L_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) = L\left(M^*_{t+1}\right) - E_t(\eta_{t+1}).$$

Likewise, one can show that:

$$L_t\left(M_{t+1}\exp(-\eta_{t+1})\right) = L\left(M_{t+1}\right) + E_t(\eta_{t+1}).$$

Finally, we derive restrictions the set of feasible $\mu_{t,\eta}$ from non-negativity of $L_t\left(M_{t+1}\exp(-\eta_{t+1})\right)$, $L_t\left(M^*_{t+1}\exp(\eta_{t+1})\right)$ and $L_t\left(\frac{S_{t+1}}{S_t}\right)$. To start, note that:

$$L_t\left(M_{t+1}\exp(-\eta_{t+1})\right) = \log E_t\left(M_{t+1}\exp(\eta_{t+1})\right) - E_t\log\left(M_{t+1}\right) + E_t(\eta_{t+1}) \geq 0$$

$$L_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) = \log E_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) - E_t\log\left(M^*_{t+1}\right) - E_t(\eta_{t+1}) \geq 0$$

This implies that the following restrictions need to be satisfied:

$$-\mu_{t\eta} \leq \log E_t\left(M_{t+1}\exp(-\eta_{t+1})\right) - E_t\log\left(M_{t+1}\right).$$

$$\mu_{t\eta} \leq \log E_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) - E_t\log\left(M^*_{t+1}\right),$$

which in turn implies that:

$$-\left(\log E_t\left(M_{t+1}\exp(-\eta_{t+1})\right) - E_t\log\left(M_{t+1}\right)\right) \leq \mu_{t\eta} \leq \log E_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) - E_t\log\left(M^*_{t+1}\right)$$

Finally, we also know that

$$L_t\left(\frac{S_{t+1}}{S_t}\right) = -E_t(\eta_{t+1}) + \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right) - E_t \log\left(\frac{M^*_{t+1}}{M_{t+1}}\right) \geq 0$$

This, in turn, implies that:

$$\log E_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right) - E_t \log\left(\frac{M^*_{t+1}}{M_{t+1}}\right) \geq \mu_{t\eta}$$

□

Proof of Corollary 4:

Proof. Note that the entropy of the ratio of two random variables is:

$$\begin{aligned} L_t\left(\frac{X_{t+1}}{Y_{t+1}}\right) &= \log E_t\left(\frac{X_{t+1}}{Y_{t+1}}\right) - E_t(\log X_{t+1}) + E_t(\log Y_{t+1}) \\ &= \log E_t\left(\frac{X_{t+1}}{Y_{t+1}}\right) + L_t(X_{t+1}) - \log E_t(X_{t+1}) - L(Y_{t+1}) + \log E_t(Y_{t+1}). \end{aligned}$$

By applying this fomula to the following expression with $X_{t+1} = M^*_{t+1}/M_{t+1}$ and $Y_{t+1} = M^*_{t+1}/[M_{t+1}e^{-\eta_{t+1}}]$, we obtain

$$\begin{aligned} L_t\left(e^{-\eta_{t+1}}\right) &= L_t\left(\frac{M^*_{t+1}/M_{t+1}}{M^*_{t+1}/[M_{t+1}e^{-\eta_{t+1}}]}\right) \\ &= L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) - L_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right) + \log E_t\left(e^{-\eta_{t+1}}\right) - \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) + \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right), \end{aligned}$$

This last step leads to the result in the text as the second term is the entropy of the change in exchange rates.

□

Proof of Corollary 5:

Proof. The first result just follows from the definition of the log change in the exchange rate and the definition of the risk-free rate at home and abroad. The second result follows immediately because $E_t[rx^{FX}_{t+1}] + L_t(rx^{FX}_{t+1}) = E_t[rx^{FX}_{t+1}] + L_t(S_{t+1}/S_t)$; only S_{t+1}/S_t is random.

$$\begin{aligned} \Delta RP &= RP^{IM} - RP^{CM} = -L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) + \mu_{t,\eta} + L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) \\ &= -L_t\left(e^{-\eta_{t+1}}\right) + \log E_t\left(e^{-\eta_{t+1}}\right) - \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) + \log E_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) + \mu_{t,\eta} \\ &= -\log E_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) + \log E_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) = \Delta L + \mu_{t,\eta}. \end{aligned}$$

The second line uses the entropy of a ratio of two random variables. □

Proof of Result 4:

Proof. We start from the complete market benchmark. The conditional entropy of the pricing kernel M_{t+1} is equal to:

$$\begin{aligned} L_t(M_{t+1}) &= L_t\left(e^{-\gamma\Delta c_{t+1}}\right) = L_t\left(e^{-\gamma w_{t+1}}\right) + L_t\left(e^{-\gamma z_{t+1}}\right) \\ &= \frac{\gamma^2\sigma^2}{2} + \varpi\left(e^{-\gamma\theta+(\gamma\delta)^2/2} - 1\right) + \gamma\varpi\theta. \end{aligned}$$

The entropy of the jump component is presented in Equation (24), page 1981 of Backus, Chernov, and Zin (2011) and derived in their Appendix A. The entropy of the 'complete spanning' exchange rate is given by:

$$\begin{aligned}
L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) &= L_t\left(e^{-\gamma(\Delta c^*_{t+1}-\Delta c_{t+1})}\right) = L_t\left(e^{-\gamma w^*_{t+1}}\right) + L_t\left(e^{-\gamma z^*_{t+1}+\gamma z_{t+1}}\right) + L_t\left(e^{\gamma w_{t+1}}\right),\\
&= \frac{\gamma^{2,*}\sigma^{*,2}}{2} + \frac{\gamma^2\sigma^2}{2} + \varpi\left(e^{-\gamma\theta^*+\gamma\theta-\gamma\gamma^*\rho_{z,z^*}\delta\delta^*+(\gamma\delta)^2/2+(\gamma\delta^*)^2/2} - 1\right) + \gamma^*\varpi\theta^* - \gamma\varpi\theta.
\end{aligned}$$

The log currency risk premium is given by the difference in the entropy of the domestic and the foreign pricing kernels:

$$\begin{aligned}
E_t\left[rx^{FX}_{t+1}\right] &= -L_t(M^*_{t+1}) + L_t(M_{t+1}) = -L_t(e^{-\gamma\Delta c^*_{t+1}}) + L_t(e^{-\gamma\Delta c_{t+1}}),\\
&= -L_t(e^{-\gamma w^*_{t+1}}) - L_t(e^{-\gamma z^*_{t+1}}) + L_t(e^{-\gamma w_{t+1}}) + L_t(e^{-\gamma z_{t+1}}),\\
&= -\frac{\gamma^{2,*}\sigma^{*,2}}{2} - \varpi\left(e^{-\gamma\theta^*+(\gamma\delta^*)^2/2} - 1\right)\\
&+ \frac{\gamma^2\sigma^2}{2} + \varpi\left(e^{-\gamma\theta+(\gamma\delta)^2/2} - 1\right) - (\gamma^*\varpi\theta^* - \gamma\varpi\theta).
\end{aligned}$$

Hence, the foreign currency risk premium in levels is given by:

$$\begin{aligned}
E_t\left[rx^{FX}_{t+1}\right] + L_t\left[rx^{FX}_{t+1}\right] &= \gamma^2\sigma^2 + \varpi\left(e^{-\gamma\theta+(\gamma\delta)^2/2} - 1\right) - \varpi\left(e^{-\gamma\theta^*+(\gamma\delta^*)^2/2} - 1\right)\\
&+ \varpi\left(e^{-\gamma\theta^*+\gamma\theta-2\gamma\gamma^*\rho_{z,z^*}\delta\delta^*+(\gamma\delta)^2/2+(\gamma\delta^*)^2/2} - 1\right).
\end{aligned}$$

Next, we introduce incomplete spanning as described in the main text. The conditional entropy of the perturbed pricing kernel is equal to:

$$\begin{aligned}
L_t\left(M_{t+1}e^{-\eta_{t+1}}\right) &= L_t\left(e^{-\gamma\Delta c_{t+1}-\gamma e_{t+1}}\right) = L_t\left(e^{-\gamma w_{t+1}}\right) + L_t\left(e^{-\gamma z_{t+1}-\gamma d_{t+1}}\right),\\
&= \gamma^2\sigma^2/2 + \varpi\left(e^{-\gamma(\theta+\theta_d)+\gamma^2\delta\delta_d\rho_{z,d}+(\gamma\delta_d)^2/2+(\gamma\delta)^2/2} - 1\right) + \gamma\varpi(\theta+\theta_d)
\end{aligned}$$

The entropy of the sum of two Poisson mixtures ($L_t\left(e^{-\gamma z_{t+1}-\gamma d_{t+1}}\right)$ above) is a generalization of the result presented in Backus, Chernov, and Zin (2011). The co-entropy condition in Proposition 2, $\mu_{t,\eta} = L_t\left(M_{t+1}e^{-\eta_{t+1}}\right) - L\left(M_{t+1}\right)$, implies here that:

$$\begin{aligned}
\gamma\varpi\theta_d &= L_t\left(M_{t+1}e^{-\eta_{t+1}}\right) - L\left(M_{t+1}\right)\\
&= \varpi\left(e^{-\gamma(\theta+\theta_d)+\gamma^2\delta\delta_d\rho_{z,e}+(\gamma\delta_d)^2/2+(\gamma\delta)^2/2} - 1\right) - \varpi\left(e^{-\gamma\theta+(\gamma\delta)^2/2} - 1\right) + \gamma\varpi\theta_d.
\end{aligned}$$

Simplifying, we obtain:

$$0 = e^{-\gamma(\theta+\theta_d)+\gamma^2\delta\delta_d\rho_{z,d}+(\gamma\delta_d)^2/2+(\gamma\delta)^2/2} - e^{-\gamma\theta+(\gamma\delta)^2/2}.$$

This leads to:

$$-\gamma(\theta+\theta_d) + \gamma^2\delta\delta_d\rho_{z,d} + (\gamma\delta_d)^2/2 + (\gamma\delta)^2/2 = -\gamma\theta + (\gamma\delta)^2/2.$$

This is equivalent to the following restriction on the wedge:

$$-\gamma\theta_d + \gamma^2\delta\delta_d\rho_{z,d} + (\gamma\delta_d)^2/2 = 0.$$

Next, we turn to the foreign pricing kernel. The conditional entropy of the perturbed pricing kernel is equal to:

$$\begin{aligned}
L_t\left(M^*_{t+1}e^{\eta_{t+1}}\right) &= L_t\left(e^{-\gamma\Delta c^*_{t+1}+\gamma d_{t+1}}\right) = L_t\left(e^{-\gamma w^*_{t+1}}\right) + L_t(e^{-\gamma z_{t+1}+\gamma d_{t+1}})\\
&= \gamma^2\sigma^{2,*}/2 + \varpi^*\left(e^{-\gamma(\theta^*-\theta^*_e)-\gamma^2\delta^*\delta_d\rho_{z^*,d}+(\gamma\delta^*_d)^2/2+(\gamma\delta^*)^2/2} - 1\right) + \gamma\varpi(\theta^*) - \gamma\varpi(\theta_d)
\end{aligned}$$

The co-entropy condition in Proposition 2, $-\mu_{t,\eta} = L_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) - L\left(M^*_{t+1}\right)$, implies here that:

$$\left[1 - e^{\gamma\theta_d - \gamma^2\delta\delta_d\rho_{z^*,d} + (\gamma\delta_d)^2/2)}\right]\varpi e^{-\gamma\theta^* + (\gamma\delta^*)^2/2} = 0.$$

This is equivalent to the following condition:

$$\gamma\theta_d - \gamma^2\delta^*\delta_d\rho_{z^*,d} + (\gamma\delta_d)^2/2 = 0.$$

Collecting all of the no-arbitrage restrictions, we obtain the conditions first described in Result 4:

$$\begin{aligned} -\gamma\theta_e + \gamma^2\delta\delta_e\rho_{z,e} + (\gamma\delta_e)^2/2 &= 0 \\ \gamma\theta_e - \gamma^2\delta^*\delta_e\rho_{z^*,e} + (\gamma\delta_e)^2/2 &= 0 \\ \gamma^2\delta\delta_e\rho_{z,e} - \gamma^2\delta^*\delta_e\rho_{z^*,e} + (\gamma\delta_e)^2/2 &= 0. \end{aligned}$$

The third condition is implied by the first two conditions.

We turn now to the entropy of the exchange rate. When markets are incomplete, the exchange rate's entropy is given by:

$$\begin{aligned} L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) &= L_t\left(e^{-\gamma\Delta c^*_{t+1} + \gamma d_{t+1} + \gamma\Delta c_{t+1}}\right), \\ &= L_t\left(e^{-\gamma w^*_{t+1}}\right) + L_t\left(e^{\gamma w_{t+1}}\right) + L_t\left(e^{-\gamma z^*_{t+1} + \gamma z_{t+1} + \gamma d_{t+1}}\right), \\ &= \frac{\gamma^2\sigma^{*,2}}{2} + \frac{\gamma^2\sigma^2}{2} + \gamma^*\varpi^*\theta^* - \gamma\varpi\theta - \gamma\varpi\theta_d \\ &\quad + \varpi\left(e^{\gamma(\theta+\theta_d-\theta^*) - \gamma^2\delta^*\delta_d\rho_{z^*,d} + \gamma^2\delta\delta_d\rho_{z,d} - \gamma^2\rho_{z,z^*}\delta\delta^* + \frac{(\gamma\delta_d)^2}{2} + \frac{(\gamma\delta)^2}{2} + \frac{(\gamma\delta^*)^2}{2}} - 1\right). \end{aligned}$$

The entropy gap between the complete and incomplete spanning exchange rate is thus:

$$\begin{aligned} L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) - L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) &= \varpi\left(e^{\gamma(\theta+\theta_d-\theta^*) - \gamma^2\delta^*\delta_d\rho_{z^*,d} + \gamma^2\delta\delta_d\rho_{z,d} - \gamma^2\rho_{z,z^*}\delta\delta^* + \frac{(\gamma\delta_d)^2}{2} + \frac{(\gamma\delta)^2}{2} + \frac{(\gamma\delta^*)^2}{2}} - 1\right) \\ &- \gamma\varpi\theta_d - \varpi\left(e^{-\gamma\theta^* + \gamma\theta - \gamma^2\rho_{z,z^*}\delta\delta^* + (\gamma\delta)^2/2 + (\gamma\delta^*)^2/2} - 1\right) \end{aligned}$$

Using the no-arbitrage condition on the wedges $\gamma\theta_d = \gamma^2\delta^*\delta_d\rho_{z^*,d} - (\gamma\delta_d)^2/2 = 0$, we obtain the following result:

$$\begin{aligned} L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) - L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) &= \varpi\left(e^{\gamma(\theta-\theta^*) + \gamma^2\delta\delta_d\rho_{z,d} - \gamma^2\rho_{z,z^*}\delta\delta^* + \frac{(\gamma\delta)^2}{2} + \frac{(\gamma\delta^*)^2}{2}} - 1\right) \\ &- \gamma\varpi\theta_d\varpi\left(e^{-\gamma\theta^* + \gamma\theta - \gamma^2\rho_{z,z^*}\delta\delta^* + \frac{(\gamma\delta)^2}{2} + \frac{(\gamma\delta^*)^2}{2}} - 1\right). \end{aligned}$$

This can be restated as :

$$\begin{aligned} \Delta L_t = L_t^{IM} - L_t^{CM} &= L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) - L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) \\ &= -\gamma\varpi\theta_d + \varpi\left(e^{-\gamma\theta^* + \gamma\theta - \gamma^2\rho_{z,z^*}\delta\delta^* + \frac{(\gamma\delta)^2}{2} + \frac{(\gamma\delta^*)^2}{2}}\right)(e^{\gamma^2\delta\delta_d\rho_{z,d}} - 1). \end{aligned}$$

This is the second part of Result 4. Taking into account the no-arbitrage conditions on the wedge, when the wedge does not have a drift ($\theta_d = 0$) and the two countries share the same parameters ($\theta = \theta^*$, $\delta = \delta^*$), we obtain:

$$\Delta L_t = \varpi\left(e^{-\gamma^2\rho_{z,z^*}\delta^2 + (\gamma\delta)^2}\right)(e^{(-\gamma^2\delta_e^2} - 1) < 0.$$

Finally, we turn to the risk premium in levels on a long position in foreign currency, which is given by :

$$E_t\left[rx^{FX}_{t+1}\right] + L_t\left(\frac{S_{t+1}}{S_t}\right) = L_t\left(M_{t+1}\right) - L_t\left(M^*_{t+1}\right) + \mu_{t,\eta} + L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right).$$

Hence, the change in the risk premium from complete to incomplete spanning is given by the change in entropy, $L_t^{IM} - L_t^{CM}$, plus the drift term: $\gamma\varpi\theta_d$. As a result, the change in the risk premium is given by:

$$\Delta RP_t = RP_t^{IM} - RP_t^{CM} = \varpi\left(e^{-\gamma\theta^* + \gamma\theta - \gamma^2\rho_{z,z^*}\delta\delta^* + \frac{(\gamma\delta)^2}{2} + \frac{(\gamma\delta^*)^2}{2}}\right)(e^{\gamma^2\delta\delta_d\rho_{z,d}} - 1).$$

This is the third part of Result 4. □

Proof of Result 5:

Proof. We need to implement the following conditions:

$$\begin{aligned} covar_t\left(m_{t+1}^*, \eta_{t+1}\right) &= -\mu_{t,\eta} - \frac{1}{2}var_t\left(\eta_{t+1}\right), \\ covar_t\left(m_{t+1}, \eta_{t+1}\right) &= -\mu_{t,\eta} + \frac{1}{2}var_t\left(\eta_{t+1}\right), \end{aligned}$$

Using the expression for the SDF, we obtain the following conditions:

$$\begin{aligned} -\sqrt{\gamma^*}\sqrt{(\gamma^* - \lambda^*)}z_t^* &= -(\psi z_t + \psi^* z_t^*) - \frac{1}{2}\left((\gamma - \kappa)z_t + (\gamma^* - \kappa^*)z_t^*\right), \\ +\sqrt{\gamma}\sqrt{(\gamma - \lambda)}z_t &= -(\psi z_t + \psi^* z_t^*) + \frac{1}{2}\left((\gamma - \kappa)z_t + (\gamma^* - \kappa^*)z_t^*\right). \end{aligned}$$

These conditions imply that:

$$\begin{aligned} \psi^* &= \frac{1}{2}(\gamma^* - \kappa^*), \\ \psi &= -\frac{1}{2}(\gamma - \kappa). \end{aligned}$$

as well as:

$$\begin{aligned} -\sqrt{\gamma^*}\sqrt{(\gamma^* - \lambda^*)} &= -\psi^* - \frac{1}{2}\left(\gamma^* - \kappa^*\right) = -\left(\gamma^* - \kappa^*\right), \\ +\sqrt{\gamma}\sqrt{(\gamma - \lambda)} &= -\psi + \frac{1}{2}\left(\gamma - \kappa\right) = \left(\gamma - \kappa\right), \end{aligned}$$

where we have used the expressions for the ψ's. This delivers the following end result:

$$\begin{aligned} \gamma^* - \sqrt{\gamma^*}\sqrt{(\gamma^* - \lambda^*)} &= \kappa^*, \\ \gamma - \sqrt{\gamma}\sqrt{(\gamma - \lambda)} &= \kappa. \end{aligned}$$

□

Proof of Result 6:

Proof. The risk premium in logs on a long position in foreign currency is given by:

$$\begin{aligned} E_t[rx_{t+1}^{FX}] &= r_t^{f,*} - r_t^f + E_t(\Delta s_{t+1}) = \frac{1}{2}\left[var_t\left(m_{t+1}\right) - var_t\left(m_{t+1}^* + \eta_{t+1}\right)\right] \\ &= \frac{1}{2}\left[(\gamma + 2\psi)z_t - (\gamma^* - 2\psi^*)z_t^*\right]. \\ &= \frac{1}{2}\left[(\gamma - (\gamma - \kappa))z_t - (\gamma^* - (\gamma^* - \kappa^*))z_t^*\right] \\ &= \frac{1}{2}\left[\kappa z_t - \kappa^* z_t^*\right] \end{aligned}$$

The risk premium in levels on a long position in foreign currency is given by:

$$
\begin{aligned}
E_t[rx^{FX}_{t+1}] + \frac{1}{2} var_t[rx^{FX}_{t+1}] &= -cov_t(m_{t+1}, \Delta s_{t+1}) \\
&= \frac{1}{2}\left[(\kappa + \kappa) z_t - (\kappa^* - \kappa^*) z_t^*\right] \\
&= \kappa z_t
\end{aligned}
$$

Recall that the short rate is given by: $r_t = \alpha + (\chi - \frac{1}{2}\gamma) z_t$. Hence, the regression slope coefficient on $r_t - r_t^*$ is

$$
\frac{cov(rx^{FX}_{t+1}, r_t - r_t^*)}{var(r_t - r_t^*)} = \frac{.5\kappa(\chi - \frac{1}{2}\gamma) + .5\kappa^*(\chi - \frac{1}{2}\gamma^*)}{(\chi - \frac{1}{2}\gamma)^2 + (\chi - \frac{1}{2}\gamma^*)^2}
$$

Hence, in the symmetric case, we end up with:

$$
\frac{.5\kappa}{(\chi - \frac{1}{2}\gamma)}
$$

□

B Quantitative Implications in Asymmetric Models

In this section, we study the case of asymmetric models, where the volatilities of the log home and foreign SDFs differ. In the main text, we assume that $std_t(m_{t+1}) = std_t(m^*_{t+1}) = 0.5$. In this appendix, we assume that $std_t(m_{t+1}) = 0.54$ and $std_t(m^*_{t+1}) = 0.46$. Since the average volatility of the two SDFs is the same as in the benchmark case, the volatility of the wedge needed to match the empirical volatility of the exchange rates is also the same as in the main text. We thus focus on the currency risk premium and the exchange rate cyclicality.

Figure 7 plots the theoretical currency risk premium in logs and levels and its empirical counterpart. The parameters are identical to those in Figure 1, matching an exchange rate volatility of 11%. The currency risk premia are plotted against the first moment of the wedge, $E_t(\eta_{t+1})$. The key difference with the main text is that the complete market model delivers a large currency risk premium, since in that case

$$
E_t[rx^{FX}_{t+1}] = \frac{1}{2}\left[var_t\left(m_{t+1}\right) - var_t\left(m^*_{t+1}\right)\right] = 4\%.
$$

One does not need to add any exchange rate predictability (through the first moment of the wedge) in order to match the currency risk premium in our sample. Matching a larger currency risk premium would call for more asymmetry in the volatilities of m_{t+1} and m^*_{t+1} because the range of permissible drifts is limited.

Figure 8 plots different measures of exchange rate cyclicality against the drift of the wedge. The parameters are the same as for Figure 7, where the volatility of the wedge is chosen to match the volatility of the exchange rate changes. The difference with Figure 3 in the main text is twofold. First, when markets are complete, the correlation between the home SDF and the change in exchange rates is even more negative than before: it is now close to -0.8, implying a strong appreciation of the home currency in bad times at home. Second, even when introducing a large drift in the wedge, this correlation is still less than 0.1, and thus never imply a strong depreciation of the home currency in bad times at home. Our key cyclicality result, the Backus-Smith slope of one, is naturally unchanged.

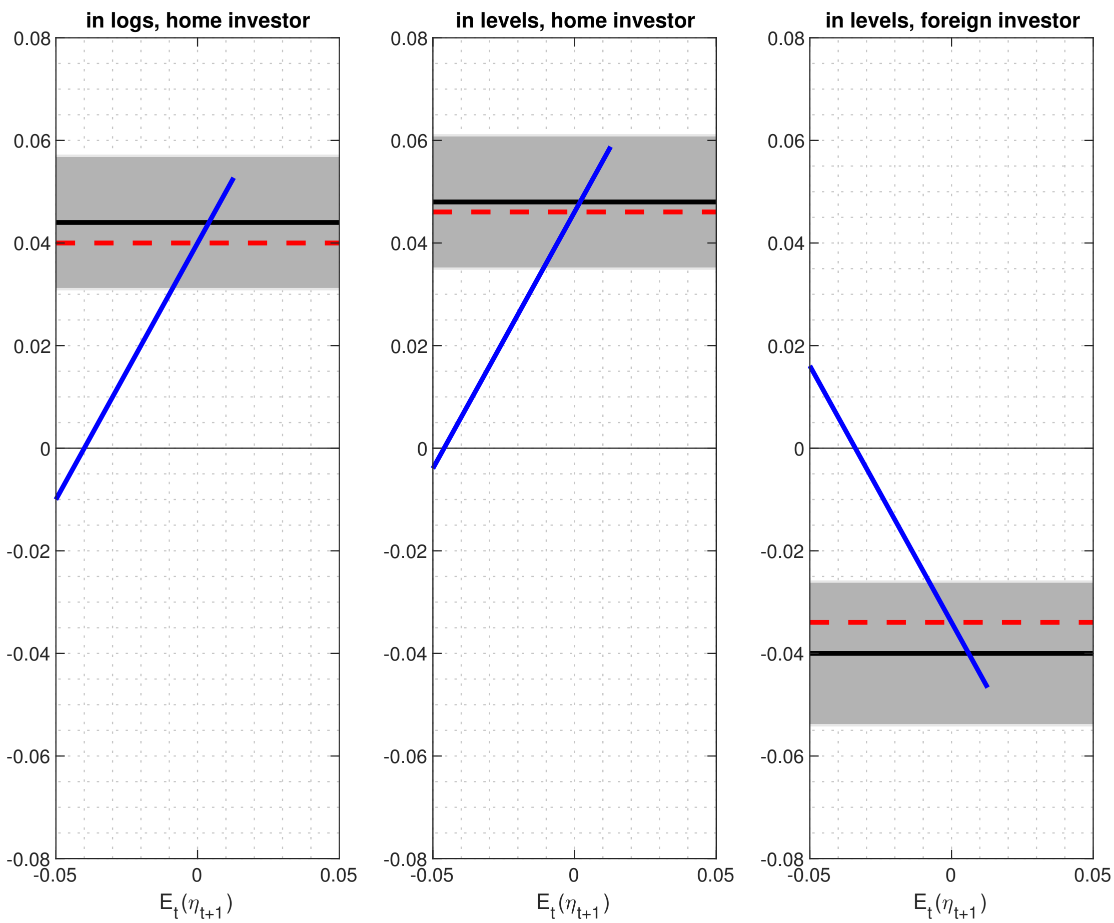

Figure 7: Currency Risk Premia — Asymmetric Case: The figure reports the foreign currency risk premium in logs (left panel), as well as in levels, from the perspective of the home investor (center panel) or foreign investor (right panel), against the first moment of the incomplete market wedge, denoted $E_t(\eta_{t+1})$. The figure is drawn assuming a maximum Sharpe ratio of 0.54 and 0.46 in the home and foreign countries ($std_t(m_{t+1}) = 0.54$ and $std_t(m^*_{t+1}) = 0.46$). The volatility of the wedge, $std_t(\eta_{t+1})$, is chosen to match the empirical volatility of the exchange rate changes (11%). The red dotted line shows each moment in a complete market model with the same SDF volatilities. The gray area indicates the value of the average carry trade excess return in the data: it is centered around the mean log excess return (4.4%, left panel) or the mean excess return from the perspective of the home and foreign investor (4.8% and −4.0% in the center and right panels); the area represents one standard error (1.3%) above and below the mean.

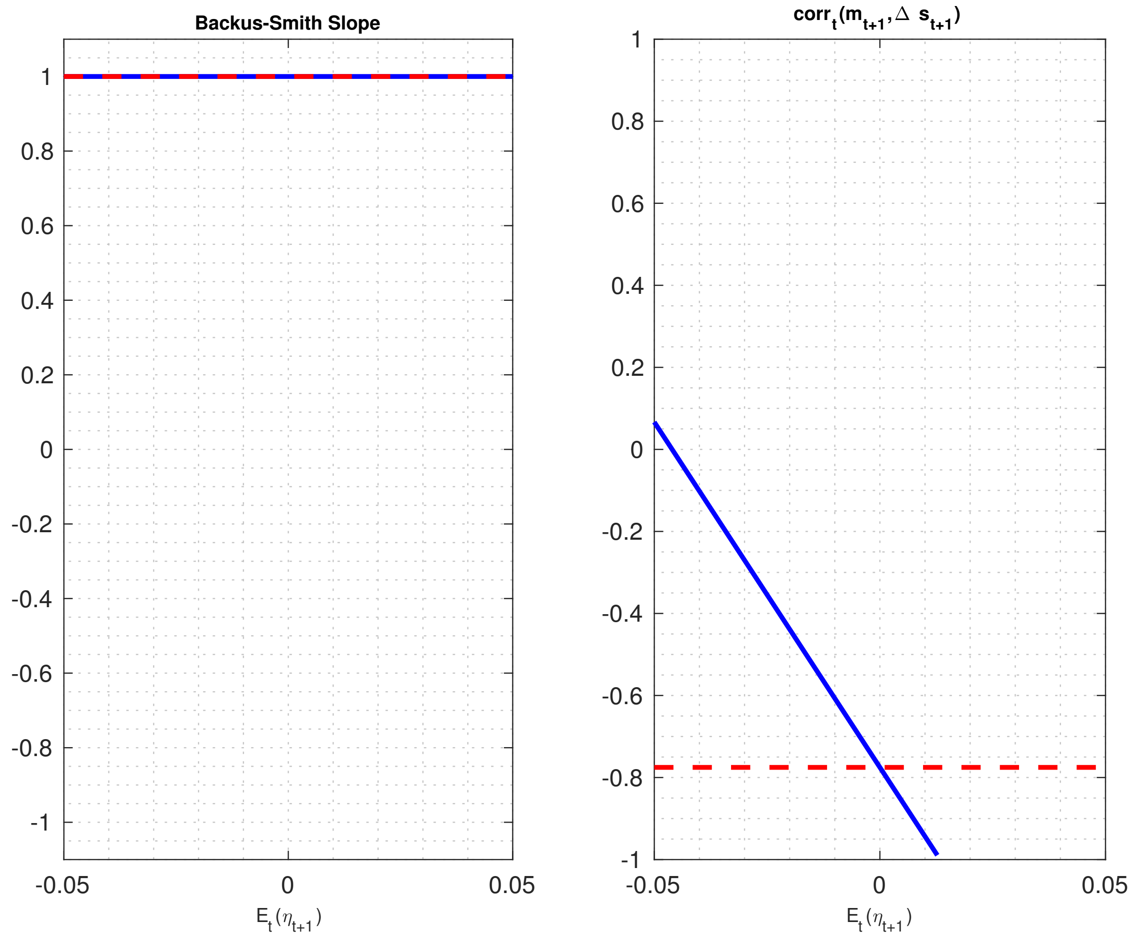

Figure 8: Exchange Rate Cyclicality — Asymmetric Case: The figure reports the slope coefficient in a regression of the difference in log SDFs, $m^*_{t+1} - m_{t+1}$ on the log change in exchange rates (left panel) and the correlation between the log home SDF and the change in the exchange rates, $corr_t(\Delta s_{t+1}, m_{t+1})$, (left panel) against the first moment of the incomplete market wedge, denoted $E_t(\eta_{t+1})$. The red dotted line shows the values of these three moments when markets are complete. The figure is drawn assuming a maximum Sharpe ratio of 0.54 and 0.46 in the home and foreign countries ($std_t(m_{t+1}) = 0.54$ and $std_t(m^*_{t+1}) = 0.46$).

C Three Examples

This section presents three examples: a simple consumption-based example; a Cox, Ingersoll, and Ross (1985) model with common factors; a consumption-based example with heteroscedasticity.[12]

C.1 A Simple Consumption-Based Example

In this section, we study in detail the consumption-based example that is mentioned rapidly in the main text.

Complete Markets We start from the complete market benchmark. The model is described in the main text.

Result 7. *The complete markets foreign currency risk premium in levels (defined from the perspective of the home investor) is given by:*

$$E_t\left[rx_{t+1}^{FX}\right] + L_t\left[rx_{t+1}^{FX}\right] = \gamma^2\sigma^2 - \gamma^2\rho_{w,w^*}\sigma\sigma^*.$$

The proof of Result 7 is as follows.

Proof. The entropy of the domestic pricing kernel is given by:

$$L_t(M_{t+1}) = L_t\left(e^{-\gamma\Delta c_{t+1}}\right) = \frac{\gamma^2\sigma^2}{2}.$$

As a result, the entropy of the exchange rate is:

$$L_t\left(\frac{M_{t+1}^*}{M_{t+1}}\right) = L_t(e^{-\gamma w_{t+1}^* + \gamma w_{t+1}}) = \frac{\gamma^2\sigma^{*2}}{2} + \frac{\gamma^2\sigma^2}{2} - \gamma^2\rho_{w,w^*}\sigma\sigma^*.$$

When markets are complete, the log currency risk premium is given by the difference in the entropy of the domestic and the foreign pricing kernels:

$$\begin{aligned} E_t\left[rx_{t+1}^{FX}\right] &= -L_t\left(M_{t+1}^*\right) + L_t(M_{t+1}) = -L_t(e^{-\gamma\Delta c_{t+1}^*}) + L_t(e^{-\gamma\Delta c_{t+1}}) \\ &= -L_t(e^{-\gamma w_{t+1}^*}) + L_t(e^{-\gamma w_{t+1}}) = -\frac{\gamma^{2,*}\sigma^{*,2}}{2} + \frac{\gamma^2\sigma^2}{2}. \end{aligned}$$

As a result, the currency risk premium in levels (defined from the perspective of the home investor) is given by:

$$E_t\left[rx_{t+1}^{FX}\right] + L_t\left[\frac{S_{t+1}}{S_t}\right] = \gamma^2\sigma^2 - \gamma^2\rho_{w,w^*}\sigma\sigma^*.$$

Likewise, the currency risk premium in levels (defined from the perspective of the foreign investor) is given by:

$$-E_t\left[rx_{t+1}^{FX}\right] + L_t\left[\frac{S_t}{S_{t+1}}\right] = \gamma^2\sigma^{*2} - \gamma^2\rho_{w,w^*}\sigma\sigma^*.$$

□

Incomplete markets Next, we introduce incomplete spanning. Assume that the wedge takes the form $\eta_{t+1} = \gamma d_{t+1}$, where $d \sim N(\mu_d, \sigma_d^2)$.

[12]Other examples of multi-country term structure models that rely on the complete market assumption to address the carry trade and forward premium puzzle include Frachot (1996), Hodrick and Vassalou (2002), Brennan and Xia (2006), Graveline and Joslin (2011), Sarno, Schneider, and Wagner (2012), and Lustig, Roussanov, and Verdelhan (2011, 2014).

Result 8. *The wedge has to satisfy the following conditions:*

$$\begin{aligned} \mu_d &= \frac{\gamma^2\sigma_d^2}{2} + \rho_{w,d}\gamma^2\sigma\sigma_d, \\ -\mu_d &= \frac{\gamma^2\sigma_d^2}{2} - \rho_{w^*,d}\gamma^2\sigma^*\sigma_d. \end{aligned}$$

The change in exchange rate variance from complete to incomplete spanning is given by:

$$\Delta Var_t = Var_t^{IM} - Var_t^{CM} = -\gamma^2\sigma_d^2.$$

The change in the currency risk premium (defined from the perspective of the home investor) from complete to incomplete spanning is given by:

$$\Delta RP_t = RP_t^{IM} - RP_t^{CM} = \rho_{w,d}\gamma^2\sigma\sigma_d.$$

The change in the currency risk premium (defined from the perspective of the foreign investor) from complete to incomplete spanning is:

$$\Delta RP_t^* = RP_t^{*IM} - RP_t^{*CM} = -\rho_{w^*,d}\gamma^2\sigma\sigma_d.$$

Result 8 implies that in the symmetric case (when the drift of the wedge is zero), the change in the currency risk premium in level is $\Delta RP_t = \Delta RP_t^* = -.5\gamma^2\sigma_d^2$. In that case, introducing a wedge decreases the currency risk premium from the perspective of both domestic and foreign agents. The Sharpe ratio declines as well:

$$SR_t^{FX} = \frac{\gamma}{\sqrt{2}}\sqrt{\sigma^2(1-\rho) - \frac{\sigma_d^2}{2}}.$$

The proof of Result 8 is as follows:

Proof. The conditional entropy of the perturbed home pricing kernel is given by:

$$L_t\left(M_{t+1}e^{-\eta_{t+1}}\right) = L_t(e^{-\gamma\Delta c_{t+1} - \gamma d_{t+1}}) = L_t(e^{-\gamma w_{t+1} - \gamma d_{t+1}}) = \frac{\gamma^2\sigma^2}{2} + \frac{\gamma^2\sigma_d^2}{2} + \rho_{w,d}\gamma^2\sigma\sigma_d.$$

Applying Proposition 2, it then implies that the drift of the wedge satisfies:

$$\mu_e = \frac{\gamma^2\sigma_d^2}{2} + \rho_{w,d}\gamma^2\sigma\sigma_d.$$

The conditional entropy of the perturbed foreign pricing kernel is equal to:

$$L_t\left(M_{t+1}^* e^{\eta_{t+1}}\right) = L_t(e^{-\gamma\Delta c_{t+1}^* + \gamma d_{t+1}}) = L_t(e^{-\gamma w_{t+1}^* + \gamma d_{t+1}}) = \frac{\gamma^2\sigma^{2,*}}{2} + \frac{\gamma^2\sigma_d^2}{2} - \rho_{w^*,d}\gamma^2\sigma\sigma_d.$$

Proposition 2 then implies that the drift of the wedge satisfies:

$$-\mu_d = \frac{\gamma^2\sigma_d^2}{2} - \rho_{w^*,d}\gamma^2\sigma\sigma_d.$$

When markets are incomplete, the entropy of the 'incomplete spanning' exchange rate is given by:

$$\begin{aligned} L_t\left(\frac{M_{t+1}^* e^{\eta_{t+1}}}{M_{t+1}}\right) &= L_t(e^{-\gamma\Delta c_{t+1}^* + \gamma d_{t+1} + \gamma\Delta c_{t+1}}) = L_t(e^{-\gamma w_{t+1}^* + \gamma w_{t+1} + \gamma d_{t+1}}) \\ &= \frac{\gamma^2\sigma^{*,2}}{2} + \frac{\gamma^2\sigma^2}{2} - \gamma^2\sigma^*\sigma_d\rho_{w^*,d} + \gamma^2\sigma\sigma_d\rho_{w,d} - \gamma^2\rho_{w,w^*}\sigma\sigma^* + \frac{\gamma^2\sigma_d^2}{2}. \end{aligned}$$

By summing the two conditions that define the drift of the wedge, one obtains that:

$$0 = \gamma^2\sigma_d^2 + \rho_{w,d}\gamma^2\sigma\sigma_d - \rho_{w^*,d}\gamma^2\sigma\sigma_d.$$

The entropy of the 'incomplete spanning' exchange rate is thus simply:

$$L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) = \frac{\gamma^2\sigma^{*,2}}{2} + \frac{\gamma^2\sigma^2}{2} - \gamma^2\rho_{w,w^*}\sigma\sigma^* - \frac{\gamma^2\sigma_d^2}{2}.$$

The entropy gap between the complete and incomplete spanning exchange rate is then:

$$\Delta L_t = L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) - L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) = -\frac{\gamma^2\sigma_d^2}{2}.$$

According to Proposition 5, the risk premium in levels on a long position in foreign currency is given by :

$$E_t\left[rx^{FX}_{t+1}\right] + L_t\left(\frac{S_{t+1}}{S_t}\right) = L_t\left(M_{t+1}\right) - L_t\left(M^*_{t+1}\right) + \mu_{t,\eta} + L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right).$$

The change in the risk premium from complete to incomplete spanning is thus given by the change in entropy, $-.5\gamma^2\sigma_d^2$, plus the drift term, $\mu_d = 0.5\gamma^2\sigma_d^2 + \rho_{w,d}\gamma^2\sigma\sigma_d$. The difference between the risk premium in incomplete and complete markets is:

$$\Delta RP_t = RP_t^{IM} - RP_t^{CM} = \mu_{t,\eta} + L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) - L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) = \rho_{w,d}\gamma^2\sigma\sigma_d.$$

Similarly, the foreign risk premium in levels on a long position in foreign currency is given by :

$$-E_t\left[rx^{FX}_{t+1}\right] + L_t\left(\frac{S_t}{S_{t+1}}\right) = -L_t\left(M_{t+1}\right) + L_t\left(M^*_{t+1}\right) - \mu_{t,\eta} + L_t\left(\frac{M_{t+1}}{M^*_{t+1}e^{\eta_{t+1}}}\right).$$

The change in the foreign risk premium from complete to incomplete spanning is given by:

$$\Delta RP_t^* = RP_t^{*IM} - RP_t^{*CM} = -\rho_{w^*,d}\gamma^2\sigma\sigma_d.$$

Proposition 2 implies that the restrictions on the wedges are given by:

$$\begin{aligned}\mu_d &= \gamma^2\sigma_d^2/2 + \rho_{w,d}\gamma^2\sigma\sigma_d,\\ -\mu_d &= \gamma^2\sigma_e^2/2 - \rho_{w^*,d}\gamma^2\sigma^*\sigma_d.\end{aligned}$$

□

C.2 A Cox, Ingersoll, and Ross (1985) Example with Common Factors

The stylized model in the main text rules out correlation of interest rates across countries. However, the key insights carry over to a setting with correlated interest rates. To show this result, we use a CIR model with common factors. The Cox, Ingersoll, and Ross (1985) model (denoted CIR) is defined by the following two equations:

$$\begin{aligned}-\log M_{t+1} &= \alpha + \chi z_t + \varphi z_t^* + \sqrt{\gamma z_t}u_{t+1} + \sqrt{\delta z_t^*}u^*_{t+1}, &(34)\\ z_{t+1} &= (1-\phi)\theta + \phi z_t - \sigma\sqrt{z_t}u_{t+1},\\ z^*_{t+1} &= (1-\phi)\theta + \phi z_t - \sigma\sqrt{z_t^*}u^*_{t+1}, &(35)\end{aligned}$$

where $u_{t+1} \sim \mathbb{N}(0,1)$ and $u^*_{t+1} \sim \mathbb{N}(0,1)$ are i.i.d. The foreign pricing kernel is specified as in Equation (34) with the same parameters. However, the foreign country has different loadings:

$$-\log M_{t+1} = \alpha + \chi^* z_t + \varphi^* z_t^* + \sqrt{\gamma^* z_t}u_{t+1} + \sqrt{\delta^* z_t^*}u^*_{t+1}.$$

To give content to the notion that z_t is a domestic factor and z_t^* is a foreign factor, we assume that $\gamma \geq \gamma^*$ and that $\delta \leq \delta^*$: the domestic (foreign) pricing kernel is more exposed to the domestic (foreign) shock than the foreign (domestic) pricing kernel. We assume that investors can trade the domestic risk-free and at least two risky domestic assets[13], but they can only trade the foreign risk-free asset. The squared maximum SRs at home and abroad are, respectively, $var_t(m_{t+1}) = \gamma z_t + \delta z_t^*$, and $var_t(m_{t+1}^*) = \gamma^* z_t + \delta^* z_t^*$.

We denote the target volatility of the incomplete markets exchange rate can be stated as: $var_t(\Delta s_{t+1}) = \kappa z_t + \kappa^* z_t^*$. We can compute the implied volatility of the incomplete markets exchange rate process using our formula:

$$var_t(\Delta s_{t+1}) = (\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*})z_t + (\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*})z_t^* - var_t(\eta_{t+1}).$$

Then we simply choose the volatility of the noise to be equal to: $var_t(\eta_{t+1}) = (\gamma + \gamma^* - \kappa)z_t + (\delta + \delta^* - \kappa^*)z_t^*$.

Result 9. *In the CIR model with country-specific factors, we can define an exchange rate process S_t that satisfies $\Delta s_{t+1} = \eta_{t+1} + m_{t+1}^* - m_{t+1}$ with variance $var_t(\Delta s_{t+1}) = \kappa z_t + \kappa^* z_t^*$. where η_t follows:*

$$\begin{aligned}
\eta_{t+1} &= \beta + \psi z_t + \psi^* z_t^* - \sqrt{(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda)z_t}u_{t+1} + \sqrt{(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*)z_t^*}u_{t+1}^* \\
&+ \sqrt{(\lambda - \kappa)z_t}\epsilon_{t+1} + \sqrt{(\lambda^* - \kappa^*)z_t^*}\epsilon_{t+1}^*,
\end{aligned}$$

where ϵ_{t+1} and ϵ_{t+1}^* are $\sim N(0,1)$, $\kappa \leq \lambda \leq \gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*}$ and $\kappa^* \leq \lambda^* \leq \delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*}$. The drift imputed to exchange rates is given by $\mu_{t,\eta} = \beta + \psi z_t + \psi^* z_t^*$. where ϵ_{t+1} and ϵ_{t+1}^* are $\sim N(0,1)$, $\kappa \leq \lambda \leq \gamma$ and $\kappa^* \leq \lambda^* \leq \gamma^*$ satisfies:

$$\begin{aligned}
\kappa &= -(\sqrt{\gamma} + \sqrt{\gamma^*})\sqrt{(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda))} + \gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*}, \\
\kappa^* &= -(\sqrt{\delta} + \sqrt{\delta^*})\sqrt{(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*))} + \delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*}, \\
\psi &= -(1/2)(\sqrt{\gamma} - \sqrt{\gamma^*})\sqrt{\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda}, \\
\psi^* &= -(1/2)(\sqrt{\delta} - \sqrt{\delta^*})\sqrt{\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*}.
\end{aligned}$$

If we allowed domestic investors to trade two foreign risky assets, then the wedges disappear. The additional covariance restrictions in (13) imply that $\kappa = \lambda = \gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*}$ and $\kappa^* = \lambda^* = \delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*}$, because the log returns are affine in the shocks. This in turn implies that the wedges are zero ($\eta = 0$).

The proof of Result 9 is as follows:

Proof. Hence, we can write a square root process for η:

$$\begin{aligned}
\eta_{t+1} &= \beta + \psi z_t + \psi^* z_t^* - \sqrt{(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda)z_t}u_{t+1} + \sqrt{(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*)z_t^*}u_{t+1}^* \\
&+ \sqrt{(\lambda - \kappa)z_t}\epsilon_{t+1} + \sqrt{(\lambda^* - \kappa^*)z_t^*}\epsilon_{t+1}^*,
\end{aligned}$$

where ϵ_{t+1} and ϵ_{t+1}^* are $\sim N(0,1)$, $\kappa \leq \lambda \leq \gamma$ and $\kappa^* \leq \lambda^* \leq \gamma^*$. The drift imputed to exchange rates is given by $\mu_{t,\eta} = \beta + \psi z_t + \psi^* z_t^*$.

To ensure that the Euler equations for the risk-free are satisfied, we also need to implement the following conditions:

$$\begin{aligned}
covar_t\left(m_{t+1}^*, \eta_{t+1}\right) &= -\mu_{t,\eta} - \frac{1}{2}var_t\left(\eta_{t+1}\right), \\
covar_t\left(m_{t+1}, \eta_{t+1}\right) &= -\mu_{t,\eta} + \frac{1}{2}var_t\left(\eta_{t+1}\right).
\end{aligned}$$

[13]If they can trade two different longer maturity bonds, then the domestically traded assets span all of the shocks.

Using the expressions for the log SDFs and η, these expressions can be restated as follows:

$$
\begin{aligned}
- & \sqrt{\gamma^*}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}z_t - \sqrt{\delta^*}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)}z_t^* \\
= & -(\psi z_t + \psi^* z_t^*) - \frac{1}{2}\left((\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\kappa)z_t + (\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\kappa^*)z_t^*\right), \\
+ & \sqrt{\gamma}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}z_t + \sqrt{\delta}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda)}z_t^* \\
= & -(\psi z_t + \psi^* z_t^*) + \frac{1}{2}\left((\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\kappa)z_t + (\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\kappa^*)z_t^*\right).
\end{aligned}
$$

By collecting the terms in z_t and z_t^*, we obtain the following four equations that need to be solved for 4 unknowns:

$$
\begin{aligned}
-\sqrt{\gamma^*}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)} &= -(\psi) - \frac{1}{2}(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\kappa), \\
-\sqrt{\delta^*}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)} &= -(\psi^*) - \frac{1}{2}(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\kappa^*).
\end{aligned}
$$

$$
\begin{aligned}
+\sqrt{\gamma}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)} &= -(\psi) + \frac{1}{2}(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\kappa) \\
+\sqrt{\delta}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\kappa)} &= -(\psi^*) + \frac{1}{2}(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\kappa^*).
\end{aligned}
$$

By adding the 1st and 3rd, and the 2nd and 4th equation, we obtain the following expression for the drift terms:

$$
\begin{aligned}
(\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda} &= -2\psi), \\
(\sqrt{\delta}-\sqrt{\delta^*})\sqrt{\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*} &= -2\psi^*.
\end{aligned}
$$

By substituting for ψ and ψ^* in the original four equations, we obtain the following conditions:

$$
\begin{aligned}
+\sqrt{\gamma}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)} &= +\frac{1}{2}(\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)} + \frac{1}{2}(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\kappa), \\
-\sqrt{\delta^*}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)} &= +\frac{1}{2}(\sqrt{\delta}-\sqrt{\delta^*})\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)} + \frac{1}{2}(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\kappa^*).
\end{aligned}
$$

These conditions can be solved for κ and κ^*:

$$
\begin{aligned}
\kappa &= -2\sqrt{\gamma}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)} + (\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)} + (\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}), \\
\kappa^* &= +2\sqrt{\delta^*}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)} + (\sqrt{\delta}-\sqrt{\delta^*})\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)} + (\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}).
\end{aligned}
$$

These conditions imply that:

$$
\begin{aligned}
\kappa &= -(\sqrt{\gamma}+\sqrt{\gamma^*})\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}) + \gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}, \\
\kappa^* &= -(\sqrt{\delta}+\sqrt{\delta^*})\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)}) + \delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}.
\end{aligned}
$$

□

Result 10. *The risk premium in logs on a long position in foreign currency is:*

$$
\begin{aligned}
E_t[rx_{t+1}^{FX}] &= \frac{1}{2}\left[\gamma-\gamma^*-(\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda}\right]z_t \\
&+ \frac{1}{2}\left[\delta-\delta^*-(\sqrt{\delta}-\sqrt{\delta^*})\sqrt{\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*}\right]z_t^*
\end{aligned}
$$

The risk premium in levels on a long position in foreign currency is given by:

$$\begin{aligned} E_t[rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] &= \left[\gamma - \sqrt{\gamma}\sqrt{(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda)} - \sqrt{\gamma}\sqrt{\gamma^*}\right] z_t \\ &+ \left[\delta - \sqrt{\delta}\sqrt{(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*)} - \sqrt{\delta}\sqrt{\delta^*}\right] z^*_t. \end{aligned}$$

These expressions can readily be compared to the complete markets log currency risk premium, $\frac{1}{2}\left[(\gamma - \gamma^*)z_t + (\delta - \delta^*)z^*_t\right]$, and the complete markets risk premium in levels, $(\gamma - \sqrt{\gamma}\sqrt{\gamma^*})z_t + (\delta - \sqrt{\delta}\sqrt{\delta^*})z^*_t$. Clearly, this establishes that the incomplete markets risk premium in levels is always smaller than the complete markets risk premium. In addition, there is less return predictability as well.

The proof of Result 10 is as follows:

Proof. Note that the risk premium in logs is given by

$$\begin{aligned} E_t[rx^{FX}_{t+1}] &= r^{f,*}_t - r^f_t + E_t(\Delta s_{t+1}) = \frac{1}{2}\left[var_t\left(m_{t+1}\right) - var_t\left(m^*_{t+1} + \eta_{t+1}\right)\right] \\ &= \frac{1}{2}\left[(\gamma - \gamma^* + 2\psi)z_t + (\delta - \delta^* + 2\psi^*)z^*_t)\right] \\ &= \frac{1}{2}\left[\gamma - \gamma^* - (\sqrt{\gamma} - \sqrt{\gamma^*})\sqrt{\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda}\right] z_t \\ &+ \frac{1}{2}\left[\delta - \delta^* - (\sqrt{\delta} - \sqrt{\delta^*})\sqrt{\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*}\right] z^*_t \end{aligned}$$

The risk premium in levels on a long position in foreign currency is given by:

$$\begin{aligned} E_t[rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] &= -cov_t(m_{t+1}, \Delta s_{t+1}) \\ &= \frac{1}{2}\left[(\gamma - \gamma^* + 2\psi + \kappa)z_t + (\delta - \delta^* + 2\psi^* + \kappa^*)z^*_t)\right] \\ &= \frac{1}{2}\left[\gamma - \gamma^* + \kappa - (\sqrt{\gamma} - \sqrt{\gamma^*})\sqrt{\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda}\right] z_t \\ &+ \frac{1}{2}\left[\delta - \delta^* + \kappa^* - (\sqrt{\delta} - \sqrt{\delta^*})\sqrt{\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*}\right] z^*_t, \\ &= \frac{1}{2}\left[\gamma - \gamma^* - 2\sqrt{\gamma}\sqrt{(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda)} + (\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*})\right] z_t \\ &+ \frac{1}{2}\left[\delta - \delta^* + 2\sqrt{\delta^*}\sqrt{(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*)} + (\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*})\right] z^*_t \\ &= \left[\gamma - \sqrt{\gamma}\sqrt{(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda)} - \sqrt{\gamma}\sqrt{\gamma^*}\right] z_t \\ &+ \left[\delta - \sqrt{\delta}\sqrt{(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*)} - \sqrt{\delta}\sqrt{\delta^*}\right] z^*_t \end{aligned}$$

□

Result 11. *The Fama slope coefficient in a regression of log currency excess returns on $f_t - s_t = r_t - r^*_t$ is*

$$\begin{aligned} &\frac{cov(rx^{FX}_{t+1}, f_t - s_t)}{var(f_t - s_t)} \\ &= \frac{.5\left(\gamma - \gamma^* - (\sqrt{\gamma} - \sqrt{\gamma^*})\sqrt{\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda}\right)\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)}{\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)^2 + \left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)^2} \\ &+ \frac{.5\left(\delta - \delta^* - (\sqrt{\delta} - \sqrt{\delta^*})\sqrt{\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*}\right)\left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)}{\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)^2 + \left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)^2} \end{aligned}$$

In the relevant region of the parameter space, $(\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*) < 0$ and $(\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*) > 0$. Then the interest rate spread $r_t - r_t^*$ decreases (increases) when z_t increases (z_t^* decreases) –the precautionary motive dominates. This is needed to account for U.I.P. deviations in the data. As a benchmark, we note that the complete markets slope coefficient is given by:

$$= \frac{.5\,(\gamma)\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right) + .5\,(\delta)\left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)}{\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)^2 + \left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)^2}$$

Recall that $\gamma \geq \gamma^*$ and $\delta \leq \delta^*$. As a result, the first term now decreases in absolute value relative to the complete markets case. The second term decreases as well in absolute value. Even in the model with common factors, the slope coefficients in the predictability regression are pushed closer to zero by the incomplete spanning and we get closer to U.I.P.

The proof of Result 11 is as follows:

Proof. Recall that the short rate is given by: $r_t = \alpha + (\chi - \frac{1}{2}\gamma)z_t + (\phi - \frac{1}{2}\delta)z_t^*$. Hence, the regression slope coefficient on $f_t - s_t = r_t - r_t^*$ is

$$\frac{cov(rx_{t+1}^{FX}, f_t - s_t)}{var(f_t - s_t)} =$$

$$\frac{.5\left(\gamma - \gamma^* - (\sqrt{\gamma} - \sqrt{\gamma^*})\sqrt{\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda}\right)\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)}{\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)^2 + \left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)^2}$$

$$+ \quad \frac{.5\left(\delta - \delta^* - (\sqrt{\delta} - \sqrt{\delta^*})\sqrt{\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*}\right)\left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)}{\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)^2 + \left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)^2}$$

□

C.3 A Consumption-Based Example with Heteroscedasticity

To develop some economic intuition for the dynamics of these wedges, we look at a version of the two-country Lucas (1982) model with heteroskedastic consumption growth. This model produces time-varying risk premia. We use δ to denote the rate of time preference and γ to denote the coefficient of relative risk aversion. The real stochastic discount factor is thus given by:

$$\begin{aligned}
-\log M_{t+1} &= -(\log\delta - \gamma\mu_g) + \gamma\sigma_{g,t}e_{t+1}, \\
\sigma_{g,t}^2 &= (1-\phi)\theta + \phi\sigma_{g,t}^2 - \sigma_{g,t}e_{t+1}, \\
-\log M_{t+1}^* &= -(\log\delta - \gamma\mu_g) + \gamma\sigma_{g,t}^*e_{t+1}^*, \\
\sigma_{g,t}^{2,*} &= (1-\phi)\theta + \phi\sigma_{g,t}^{2,*} - \sigma_{g,t}^*e_{t+1}^*.
\end{aligned}$$

where $\Delta c_{t+1} = \mu_g + \sigma_{g,t}e_{t+1}$, and $\Delta c_{t+1} = \mu_g + \sigma_{g,t}^*e_{t+1}^*$. The consumption growth innovations $e_{t+1} \sim \mathbb{N}(0,1)$ and $e_{t+1}^* \sim \mathbb{N}(0,1)$ are i.i.d. as well as uncorrelated across countries. When markets are complete, the exchange rate variance is thus $var_t(\Delta s_{t+1}) = \gamma^2\sigma_{g,t}^2 + \gamma^{2,*}\sigma_{g,t}^{2,*}$. Domestic investors can invest in the domestic risk-free and at least one domestic risky asset (e.g. a longer maturity real zero-coupon bond), and the foreign risk-free, but they cannot invest in foreign risky assets. Hence, only the domestic shocks are spanned.

In this model, we can back out the dynamic process for the wedges that satisfy the necessary conditions of Proposition 1. It turns out that all the wedges take the form:

$$\begin{aligned}
\eta_{t+1} = \psi\sigma_{g,t} + \psi^*\sigma_{g,t}^* \quad &- \quad \sqrt{(\gamma^2 - \lambda)}\sigma_{g,t}e_{t+1} + \sqrt{(\gamma^2 - \lambda^*)}\sigma_{g,t}^*e_{t+1}^* \\
&+ \quad \sqrt{(\lambda - \kappa)}\sigma_{g,t}\epsilon_{t+1} + \sqrt{(\lambda - \kappa^*)}\sigma_{g,t}^*\epsilon_{t+1}^*.
\end{aligned}$$

where ϵ_{t+1} and ϵ_{t+1}^* are standard i.i.d. Gaussian shocks uncorrelated with the consumption growth innovations e_{t+1} and e_{t+1}^*. These shocks are the unspanned component of the exchange rate changes. The parameters κ and κ^* govern the volatility of the exchange rate when markets are incomplete: $var_t(\Delta s_{t+1}) = \kappa\sigma_{g,t}^2 + \kappa^*\sigma_{g,t}^{2,*}$. These wedges only affect exchange rates, and as a result, the returns on foreign investments. The returns on domestic investments remain unchanged.

The parameters κ and κ^* are the only two degrees of freedom in the law of motion of the wedge. The other parameters that describe the wedge are implied. The drift term (denoted $\mu_{t,\eta}$ in Proposition 1 and here equal to $\psi\sigma_{g,t} + \psi^*\sigma^*_{g,t}$) is governed by the consumption growth volatilities; it is determined by the no-arbitrage conditions, which imply that $\psi = -\frac{1}{2}(\gamma^2 - \kappa)$, and $\psi^* = \frac{1}{2}(\gamma^2 - \kappa^*)$. The unexpected component of the wedge depends on the parameters λ and λ^*, which have to satisfy the following restrictions: $\kappa \leq \lambda \leq \gamma^2$ and $\kappa^* \leq \lambda^* \leq \gamma^2$, and are implicitly defined by the following conditions: $\kappa = \gamma^2 - \sqrt{\gamma^2}\sqrt{\gamma^2 - \lambda}$, $\kappa^* = \gamma^2 - \sqrt{\gamma^2}\sqrt{\gamma^2 - \lambda^*}$.

In this example, the domestic investor cannot invest in any foreign risky asset. If we allow the foreign investor to do so, then we need to impose the additional covariance restrictions in condition (13). These conditions imply that η is orthogonal to e_{t+1} and e^*_{t+1}, because the log return on the domestic (foreign) risky asset is affine in the domestic (foreign) innovation, which in turn implies $\kappa = \lambda = \gamma^2$ and $\kappa^* = \lambda^* = \gamma^{*,2}$. We are back in the case of complete markets: $\eta_{t+1} = 0$.

In the two-country Lucas (1982) model, incomplete spanning reduces the exchange rate's exposure to the consumption growth innovations. Instead, the exchange rates are now exposed to shocks uncorrelated with aggregate consumption growth in either country. In the following sections, we study the impact of incomplete spanning on each of the key three exchange rate puzzles without restricting ourselves to the Lucas (1982) model. We start with the Brandt, Cochrane, and Santa-Clara (2006) puzzle.

The two-country Lucas (1982) model with heteroskedastic consumption growth provides a simple laboratory for understanding the effects of incompleteness. In that model, the complete markets risk premium in logs on a long position in foreign currency is: $E_t[rx^{FX}_{t+1}] = \frac{1}{2}\gamma^2\left[\sigma^2_{g,t} - \sigma^{2,*}_{g,t}\right]$, while the complete markets risk premium in levels is given by: $E_t[rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] = \gamma^2\sigma^2_{g,t}$. In the incomplete spanning economy, the risk premium in logs on a long position in foreign currency is $E_t[rx^{FX}_{t+1}] = \frac{1}{2}\kappa\left[\sigma^2_{g,t} - \sigma^{2,*}_{g,t}\right]$, while the risk premium in levels on a long position in foreign currency is given by: $E_t[rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] = \kappa\sigma^2_{g,t}$. The incomplete markets model behaves as if risk aversion γ was effectively reduced to $\sqrt{\kappa}$. There is also less return predictability in the incomplete spanning economy. The Fama slope coefficient in a regression of log currency excess returns on $f_t - s_t = r_t - r^*_t$ is $-2\kappa/\gamma^2$. Hence, the slope coefficient falls below 2, its complete markets value, in absolute value. The percentage reduction in the slope coefficient is twice the percentage reduction in volatility $2\log(\sqrt{\frac{\kappa}{\gamma^2}})$.

D Exchange Rate Entropy

Table 2 reports summary statistics on exchange rate entropy. At the quarterly frequency, the entropy and half-volatility are essentially the same, as if exchange rate changes were normally distributed.

Table 2: Exchange Rate Entropy

	Cross-country Mean	Cross-country Std	Cross-country Min	Cross-country Max
$L(\Delta s)$	0.64	0.15	0.19	0.80
	(0.05)	(0.02)	(0.03)	(0.08)
$\frac{1}{2}\sigma^2_{\Delta s}$	0.64	0.15	0.19	0.81
	(0.05)	(0.02)	(0.03)	(0.08)
$L(\Delta q)$	0.63	0.15	0.19	0.81
	(0.05)	(0.02)	(0.03)	(0.08)
$\frac{1}{2}\sigma^2_{\Delta q}$	0.63	0.16	0.19	0.82
	(0.05)	(0.02)	(0.03)	(0.08)

Notes: The table reports summary statistics on exchange rate entropy and volatility. The entropy, denoted $L(\Delta s)$, is measured as the log of the mean change in exchange rate minus the mean of the log change in exchange rate: $L(\Delta s) = logE\left(e^{\Delta s}\right) - E(log\Delta s)$. The volatility is measured as half the variance of the log change in exchange rates. Similar moments are defined for real exchange rates. The table presents the cross-country mean of the bilateral nominal and real exchange rate volatilities, along with the cross-country standard deviation of the bilateral exchange rate volatilities and the corresponding minimum and maximum values across countries. Similar statistics are reported for entropies. Moments are annualized (multiplied by 4) and reported in percentages. Data are quarterly, over the 1973.IV – 2014.IV period. The panel consists of 15 countries: Australia, Belgium, Canada, Denmark, France, Germany, Italy, Japan, Netherlands, New Zealand, Norway, Sweden, Switzerland, U.K., and U.S. The standard errors (reported between brackets) were generated by block-bootstrapping 10,000 samples, each block containing 2 quarters.

E Projection Arguments

We can project the respective SDFs on the space of traded assets at home and abroad. The space of internationally traded assets only includes the domestic and the foreign risk-free. We can recover our results, including the risk premium, using the projection of the SDFs onto the space of traded payoffs.[14] The usual intuition is that one can add some noise that is unspanned to the SDFs without changing the pricing implications. That intuition is false in this setting, because the noise itself changes the space of traded payoffs through its effect on exchange rates.

E.1 Projection Argument with log SDFs

We use lowercases to denote logs. When projecting the log domestic SDF onto a constant and the innovation in the log exchange rate, we get the following expression:

$$\lambda_{t+1} = proj(m_{t+1}|X) = E_t(m) + \beta\left(\Delta s_{t+1} - E(\Delta s_{t+1})\right).$$

As before, we introduce a wedge in the spot exchange rate η:

$$\Delta s_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1}.$$

Hence, the projection slope coefficient is given by:

$$\beta(\eta) = \frac{cov_t(m^*_{t+1} - m_{t+1} + \eta_{t+1}, m_{t+1})}{var_t(\Delta s_{t+1})}.$$

Similarly, when projecting the log foreign SDF onto the space of internationally traded assets, we get the following result:

$$\lambda^*_{t+1} = proj(m^*_{t+1}|X) = E_t(m^*) + \beta^*\left(-\Delta s_{t+1} + E(\Delta s_{t+1})\right).$$

Hence, the foreign projection coefficient is given by:

$$\beta^*(\eta) = -\frac{cov_t(m^*_{t+1} - m_{t+1} + \eta_{t+1}, m^*_{t+1})}{var_t(\Delta s_{t+1})}.$$

After some algebra, we obtain that the domestic projection coefficient satisfies:

$$\begin{aligned}
\beta(\eta) &= \frac{cov_t(m^*_{t+1} - m_{t+1} + \eta_{t+1}, m_{t+1})}{var_t(\Delta s_{t+1})}, \\
&= \frac{cov_t(m^*_{t+1} - m_{t+1}, m_{t+1})}{var_t(\Delta s_{t+1})} + \frac{-\mu_{t,\eta} + \frac{1}{2}var_t(\eta_{t+1})}{var_t(\Delta s_{t+1})},
\end{aligned}$$

where we have used our second condition in Proposition 1. Note that $\beta(\eta) \leq 0$. The wedge η does not drop out when we project the SDF onto the space of traded assets. Instead, the wedge determines the slope coefficient in the projection. Similarly, the foreign projection slope coefficient is given by:

$$\begin{aligned}
\beta^*(\eta) &= -\frac{cov_t(m^*_{t+1} - m_{t+1} + \eta_{t+1}, m^*_{t+1})}{var_t(\Delta s_{t+1})}, \\
&= -\frac{cov_t(m^*_{t+1} - m_{t+1}, m^*_{t+1})}{var_t(\Delta s_{t+1})} - \frac{-\mu_{t,\eta} - \frac{1}{2}var_t\left(\eta_{t+1}\right)}{var_t(\Delta s_{t+1})},
\end{aligned}$$

where we have used our first condition in Proposition 1. The wedge η does not drop out when we project the foreign SDF on the space of foreign traded assets. Note that $\beta^*(\eta) \leq 0$. Also, note that $\beta^*(\eta) + \beta^*(\eta) = -1$.

When we use these projections of the SDFs, our results are unchanged. The foreign currency risk premium for the home investor is the same as before:

$$\begin{aligned}
E_t[rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] &= -cov_t(\lambda_{t+1}, \Delta s_{t+1}) = -\beta(\eta)var_t(\Delta s_{t+1}), \\
&= var_t\left(m_{t+1}\right) - covar_t\left(m^*_{t+1}, m_{t+1}\right) - \frac{1}{2}var_t\left(\eta_{t+1}\right) + E_t\left(\eta_{t+1}\right),
\end{aligned}$$

[14]The authors acknowledge helpful conversations with John Cochrane, Bob Hodrick and Ben Hebert on this topic.

where we have used the expression for λ_{t+1}. Similarly, the currency risk premium for the foreign investor is

$$\begin{aligned} E_t[-rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] &= -cov_t(\lambda^*_{t+1}, \Delta s_{t+1}) = -\beta^*(\eta) var_t(\Delta s_{t+1}), \\ &= var_t\left(m^*_{t+1}\right) - covar_t\left(m^*_{t+1}, m_{t+1}\right) - \frac{1}{2}var_t\left(\eta_{t+1}\right) - E_t\left(\eta_{t+1}\right). \end{aligned}$$

Hence, the expressions for the risk premia are identical when we use the projections. It is not the case that the incompleteness wedge disappears from the risk premium expression.

E.2 Projection Argument with level SDFs

Brandt, Cochrane, and Santa-Clara (2006) argue that market incompleteness cannot help to resolve the volatility puzzle. This section explains why our results differ from those in Brandt, Cochrane, and Santa-Clara (2006). When projecting the domestic SDF onto the space of internationally traded assets, we get the following (projecting on ones and $\frac{S_{t+1}}{S_t}$):

$$\Lambda_{t+1} = proj(M_{t+1}|X) = E(M) + \beta\left(\frac{S_{t+1}}{S_t} - E(\frac{S_{t+1}}{S_t})\right).$$

On page 675, Brandt, Cochrane, and Santa-Clara (2006) define the following expression for the foreign projected SDF:

$$\Lambda^*_{t+1} = \Lambda_{t+1}\frac{S_{t+1}}{S_t},$$

which implies:

$$\Lambda^*_{t+1} = \left[E(M) + \beta\left(\frac{S_{t+1}}{S_t} - E(\frac{S_{t+1}}{S_t})\right)\right]\frac{S_{t+1}}{S_t},$$

since

$$\Lambda_{t+1} = proj(M_{t+1}|X) = E(M) + \beta\left(\frac{S_{t+1}}{S_t} - E(\frac{S_{t+1}}{S_t})\right).$$

This Λ^*_{t+1} satisfies the foreign investors' Euler equations; it is a valid SDF but Λ^*_{t+1} is not in the foreign payoff space (see quadratic exchange rate terms), and note the projection onto foreign payoff space yields no quadratic terms:

$$proj(M^*_{t+1}|X^*) = E(M^*) + \beta^*\left(\frac{S_t}{S_{t+1}} - E(\frac{S_t}{S_{t+1}})\right).$$

Λ^*_{t+1} cannot be the minimum variance SDF for the foreign investor. This explains why Brandt, Cochrane, and Santa-Clara (2006)'s argument for market incompleteness irrelevance does not apply here: their foreign pricing kernel is no longer in the space of traded payoffs (see Maurer and Tran, 2016, for a related argument).

Further, consider the domestic investors' Euler equations for investing in the risk-free note at home and abroad, evaluated with the projection:

$$\begin{aligned} E_t\left(\Lambda_{t+1}R^f_t\right) &= 1 \\ E_t\left(\Lambda_{t+1}\frac{S_{t+1}}{S_t}R^{f,*}_t\right) &= 1 \end{aligned}$$

The first Euler equation holds by construction. The second Euler equation implies that the multiplicative risk premium on FX is given by:

$$\frac{R^{f,*}_t E_t(\frac{S_{t+1}}{S_t}) - R^f_t}{R^f_t} = -R^{f,*}_t \beta var_t\left(\frac{S_{t+1}}{S_t}\right)$$

The projection argument implies that the incomplete markets risk premium is determined by the sensitivity of the SDF to exchange rate shocks β, as one would expect. As the variance of the exchange rates decreases, UIP is restored and the multiplicative risk premium is zero. Clearly, there is nothing that keeps us from pushing the variance of the exchange rates to zero if we adjust $E_t(\frac{S_{t+1}}{S_t})$, holding interest rates fixed; our paper shows how to do this in a log-normal setting.