The Informational Effect of Monetary Policy and the Case for Policy Commitment *

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Abstract

I explore how asymmetric information between the central bank and the private sector changes the optimal conduct of monetary policy. I build a New Keynesian model in which private agents have imperfect information about underlying shocks, while the central bank has perfect information. In this environment, private agents exact information about underlying shocks from the interest-rate decisions by the central bank. The informational effect of monetary policy makes policy changes self-defeating: when the central bank adjusts the interest rate to offset the effect caused by underlying shocks, the interest rate also reveals information about the realization of underlying shocks. This informational effect makes the equilibrium in the private sector deviate further away from the central bank's target. I show that committing to a state-contingent policy rule alleviate this self-defeating problem by controlling the information revealed through the interest rate.

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1 Introduction

When the private sector has imperfect information about underlying shocks, monetary policy has real effects. Much progress has been made to study how optimal monetary policy should change, taking as given how expectations about underlying shocks are different from the ones under full information. Yet, as recent empirical papers show, expectations about the underlying shocks in the private sector are also influenced by changes in monetary policy.[1] When expectations about underlying shocks are endogenous to policy decisions, how should the optimal monetary policy change?

To address this problem, I build a New Keynesian model with informational frictions in the private sector. Private agents have partial information on the realization of underlying shocks, whereas the central bank has perfect information. In this environment, monetary policy has dual effects: the first is the direct effect on the borrowing cost of households, and the second is the informational effect on expectations in the private sector about underlying shocks. Suppose that the economy is hit by a positive cost-push shock which is partially observed in the private sector: the central bank now faces a trade-off when making policy decisions. Without any adjustment of the interest rate, inflation increases due to the cost-push shock. If the central bank increases the interest rate to offset the effect of the cost-push shock on inflation, the direct effect reduces inflation. At the same time, however, the private sector gets information about the cost-push shock from the changes in the interest rate. This informational effect induces firms to further increase their prices, defeating the direct effect of the tightening monetary policy.

In this paper, I show that there are gains from committing to a state-contingent policy rule, which reduces the self-defeating effect due to informational frictions. The notion of gains from commitment has a long tradition, which dates back to Kydland and Prescott (1977) and Barro and Gordon (1983). The gains from commitment in these papers come from the assumption that the private sector does not know the policy itself. In this paper, I propose a new mechanism that leads to the gains from committing to a policy rule: when the private sector has perfect information about monetary policy, but has imperfect information about the underlying shocks to which the monetary policy is responding, the central bank can control the informational effect of monetary policy by committing to a policy rule.

To model asymmetric information between the central bank and the private sector, I introduce informational frictions to an otherwise canonical New Keynesian model with Calvo price rigidity. There are two types of shocks in the private sector: technology shocks and wage markup shocks. Their aggregate components translate into natural-rate shocks in the output gap and cost-push

[1] See Campbell et al. (2012) and Nakamura and Steinsson (2013) as examples of empirical studies on the informational effect of monetary policy.

shocks in inflation. Private agents, both the household and all firms, know the distribution of the shocks, but have partial information on the realization of shocks.

The central bank is assumed to have perfect information. It sets the interest rate conditional on the actual shocks to minimize the loss function which is given by the weighted sum of squared inflation and the squared output gap. Private agents knows the response function of the interest rate to different shocks, and therefore, changes in the interest rate reveal the bank's information on the realization of shocks.

The key insight of the model can be illustrated in a simple case in which underlying shocks have no serial correlation. The interest rate is one signal that jointly provides information about the two shocks. How private agents form expectations about different shocks is determined by how the interest rate responds to different shocks. Suppose that the interest rate reacts more aggressively to the natural-rate shock relative to the cost-push shock. In this situation, whenever private agents observe a change in the interest rate, they expect the change is more likely to be a response to the natural-rate shock.

A central bank can either be discretionary or commit to a policy rule. A discretionary central bank sets an optimizing interest rate at any given state of the economy and takes as given how private agents form expectations from its interest-rate decisions. I solve for the Markov perfect equilibrium between the central bank and the private sector. The private sector forms expectations while expecting the central bank to play the optimizing interest rate in equilibrium. In comparison, committing to the optimal policy rule makes expectations in the private sector less sensitive to interest rate changes, decreasing the degree to which informational effect dampens the direct effect of monetary policy.

The informational effect of monetary policy results in a novel time-inconsistency problem. Different from the traditional time inconsistency, in which the incentives to deviate apply across time periods, the time inconsistency problem in my model applies across states. Once the central bank has committed to a policy rule, it has fixed how the private sector forms expectations using the interest rate as a signal. Ex-post, the central bank has an incentive to deviate from its committed rule, assuming that such change will not change how expectations are formed in the private sector.

I extend the analysis in two ways. First, I study central bank direct communication, which is modeled by adding external signals in addition to the interest rate. The information revealed through the interest rate and the information conveyed through direct communication interact with each other. Increasing the precision of direct communication about one shock also makes the interest rate a more precise signal about the other shock. Consequently, this interaction effect yields different welfare implications from the conventional wisdom about central bank communication. Providing more precise information about the efficient shock (natural-rate shock) through central bank communication may reduce welfare, as the private sector also simultaneously gets more

precise information about the inefficient shock (cost-push shock) from the interest rate.

Second, I extend the analysis to serially correlated shocks to study the dynamic informational effect. In this case, the private agents form beliefs about current shocks by optimally weighing current signals and past beliefs. Consequently, the current interest rate has a lagged effect on future equilibrium through its effect on current beliefs. The consideration of the dynamic informational effect makes the equilibrium interest rate target beliefs in addition to targeting current inflation and the output gap.

To quantify the gains from commitment, I calibrate the full version of my model, including external signals, serially correlated shocks, and policy implementation errors. In my calibrated model, I adopt parameter values from previous macroeconomics studies, except for the precision of external information. Varying the precision of external information critically changes the size of the gains from commitment. When external signals are as precise as actual shocks, the optimal policy rule can improve welfare by 35 percent relative to the equilibrium under optimizing discretionary policy.

1.1 Related Literature

My paper connects three strands of literature: (i) the comparison of monetary policy under discretion versus commitment, (ii) optimal monetary policy under information frictions, and (iii) the informational effect of monetary policy.

(A) Discretionary Monetary Policy versus Monetary Policy Rule

There is a long history of studying the gains from monetary policy commitment. The original treatments can be found in Kydland and Prescott (1977) and Barro and Gordon (1983), who discuss the classical inflationary bias that results from a discretionary central bank having an objective function that contains a positive output gap target. A large literature has developed various methods to overcome the inflationary bias under discretion, including central bank reputation (Barro (1986) and Cukierman and Meltzer (1986) ect). and different central bank preferences (Rogoff (1985), Lohmann (1992) and Svensson (1995) etc).

Another mechanism that leads to gains from commitment is when a discretionary central bank faces stabilization bias. This occurs when there is a trade-off between closing the output gap and minimizing inflation in the current period. By committing to a delayed interest rate response, the central bank is able to decrease current inflation without sacrificing the current output gap; instead it does so through the decrease in expected future inflation. Clarida, Gali and Gertler (2000) study how an ad-hoc cost-push shock introduces a conflict between inflation stabilization and output gap stabilization and describe the optimal commitment to a future interest rate path. Woodford (1999) studies how an interest rate smoothing objective helps the central bank to commit to a history-

dependent policy, to steer private sector expectations about future policy rates. Eggertsson et al. (2003) show that optimal commitment to delayed response can mitigate the distortions created by the zero lower bound on the interest rate.

(B) Optimal Monetary Policy with Informational Frictions

My paper builds on the studies of optimal monetary policy under imperfect information. This field is revived by Woodford (2001), which shows how higher order beliefs lead to a persistent effect of monetary policy, under the assumption of imperfect information which was initially introduced in Phelps (1970) and Lucas (1972).

The majority of papers that study optimal monetary policy under informational frictions assume that beliefs in the private sector are formed independently from monetary policy decisions. Under this assumption, a central bank makes policy decisions every period, taking as given the exogenous beliefs in the private sector. Ball, Mankiw and Reis (2005) assume that information is rigid in the private sector and characterize optimal policy as an elastic price standard. Adam (2007) assumes an endogenous learning process in the private sector and demonstrates that the target of the optimal monetary policy changes from output gap stabilization to price stabilization when information becomes more precise. Angeletos and La'O (2011) solve the Ramsey problem for optimal monetary policy and show that the flexible-price equilibrium is no longer the first-best when information frictions affect real variables. Coibion and Gorodnichenko (2012) point out the lack of information in the private sector by comparing the predictability of interest rate changes made by financial market participants and made in Greenbook forecasts.

There are also papers that discuss the gains from policy commitment under imperfect information. Svensson and Woodford (2003) and Svensson and Woodford (2004) assume that the central bank has imperfect information and show that the optimal policy under commitment displays considerable inertia, relative to the discretionary policy, due to the persistence in the learning process. Lorenzoni (2010) and Paciello and Wiederholt (2013) explore the idea that the central bank is able to change the learning process in the private sector if it is able to commit to completely offset inefficient shocks.

Recent papers have begun to investigate the situation in which the private sector extracts information about the underlying economy from monetary policy decisions. Baeriswyl and Cornand (2010) note that because monetary policy cannot fully neutralize markup shocks, the central bank alters its policy response to reduce the information revealed about the cost push shock through monetary policy. Berkelmans (2011) demonstrates that with multiple shocks, tightening policy may initially increase inflation. The paper most related to the present work is Tang (2013), which shows that when the private sector has rational expectations, the stabilization bias is reduced when monetary policy has an information effect.

To the best of my knowledge, the only paper that discusses the time inconsistency problem

resulting from the informational effect of monetary policy is Stein and Sunderam (2016). The authors use a reduced-form model in which the central bank balances between implementing the optimal target rate and minimizing the information revealed about this target. In their paper, private agents are assumed not to have rational expectations about the central bank's behaviors. The discretionary central bank always has incentives to deviate from the target interest rate, to reveal less information about its target. In my paper, I assume that private agents have rational expectations about how the central bank would react under both discretionary policy and a policy rule. Relative to the perfect information case, both optimizing discretionary policy and the optimal policy rule exhibit an inertial response to cost-push shocks, but the degree of inertia is higher under commitment.

(C) Empirical Evidence on the Informational Effect of Monetary Policy

My study is motivated by the increase in the central bank transparency in the U.S. In 1994, the FOMC began to announce its the target policy rate. This change in policy is shown to improve private sector interest rate forecasts, (Swanson (2006)) and impact private forecasts of other economic fundamentals as well. Romer and Romer (2000) and Romer and Romer (2004) are the first contributions to provide empirical evidence on information asymmetry between the Federal Reserve and the private sector. They show that inflation forecasts by private agents respond to changes in the policy-rate after FOMC announcements. Faust, Swanson and Wright (2004) further confirm that the private sector revises its forecasts in response to monetary policy surprises. In more recent papers, Campbell et al. (2012) show that unemployment forecasts decrease and CPI inflation forecasts increase after a positive innovation to future federal funds rates. Nakamura and Steinsson (2013) identify the informational effect of the federal funds rate suing high-frequency data. In addition, Melosi (2016) captures this empirical pattern using a DSGE model with dispersed information. Garcia-Schmidt (2015) uses Brazilian Survey data to show that inflation forecasts in the private sector increase in the short run after an unexpected tightening policy.

The remainder of the paper is organized as follows. Section 2 characterizes the optimization decisions by the representative household in the private sector, and expresses the aggregate output gap and inflation as functions of beliefs. Section 3 analyzes the equilibrium interest rate under discretion and the optimal policy rule in the baseline case where shocks are not serially correlated. Section 4 and section 5 discuss two factors that affect the size of the gains from commitment: external information and serial correlation in shocks. To quantitatively assess the gains from commitment, I calibrate the full version of my model with serially correlated shocks, external signals and policy implementation error in Section 6. Section 7 concludes the paper.

2 Private Sector

In this section, I add informational frictions to an otherwise standard New Keynesian model with Calvo-type price rigidity. Fluctuations are driven by two types of shocks: a technology shock and a wage markup shock. I assume that the central bank has perfect information about the two shocks, whereas the private sector cannot directly observe the shocks. The private sector has rational expectations about the central bank's behaviors. In particular, the private sector correctly understands how the central bank responds to different shocks and infers information about the shocks from observing changes in the interest rate. This section solves the equilibrium output gap and inflation under imperfect information.

2.1 Information Frictions

I model an "islands economy", following similar lines as Phelps (1970), Lucas (1972), Woodford (2001), and Angeletos and La'O (2010). There is a continuum of islands, indexed by j, and informational frictions are the result of geographical isolation of islands. There is a representative household, consisting of a consumer and a continuum of workers. At the beginning of each period, each household sends one worker to each island, j. There is a continuum of monopolistic firms, each located on one island and indexed by the island. Each firm demands labor in the local labor market within the island and produces a differentiated intermediate good, j. Information is symmetric within an island, as each firm is able to observe its firm-specific shocks. Information is asymmetric across islands, as firms are unable to observe shocks or decisions made by other firms. Optimal resetting prices are strategic complements, which makes inflation to be a function of subjective beliefs in the private sector. The consumer makes inter-temporal consumption decisions. He is able to observe the current prices of all intermediate goods, but unable to directly observe the shocks. Consequently, the inter-temporal consumption decisions are also subject to informational frictions.

2.2 Private Sector Optimization Problem

2.2.1 Household

The preferences of the representative household are defined over the aggregate consumption good, C_t, and the labor supplied to each firm, $N_t(j)$, as

$$E_t^H \Sigma_{t=0}^{\infty} \beta^t \left\{ U(C_t) - \int V(N_t(j)) dj \right\}, \tag{1}$$

where E_t^H denotes the household's subjective expectations conditional on its information set, ω_H. The aggregate good C_t consists a continuum of intermediate goods:

$$C_t = \left(\int_0^1 C_t(j)^{1-\frac{1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{2}$$

where $C_t(j)$ is the consumption of intermediate good j in period t.

The economy is cashless. The household maximizes expected utility subject to the inter-temporal budget constraint:

$$\int P_t(j)C_t(j)dj + B_{t+1} \leq \int W_t(j)N_t(j)dj + (1+i_t)B_t + \Pi_t, \tag{3}$$

where B_t is a risk-free bond with nominal interest i_t, which is determined by the central bank. Π_t is the lump-sum component of household income, which includes tax payments and profits from all firms. $W_t(j)$ and $N_t(j)$ are the labor wage and labor supply for firm j, respectively.

The household's optimization problem can be solved in two stages. First, conditional on the level of aggregate consumption, the household allocates intermediate goods consumption to minimize the cost of expenditure conditional on the level of aggregate good consumption. The allocation of intermediate good consumption that minimizes expenditure yields

$$C_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon} C_t, \tag{4}$$

where $P_t = \left[\int_0^1 P_t(j)^{1-\varepsilon} dj \right]^{\frac{1}{1-\varepsilon}}$.

In the second stage, given the aggregate price level, P_t, the household chooses its aggregate consumption, C_t, labor supply to all firms, $N_t(j)$ $\forall j$, and savings in the risk-free bond, B_{t+1}. I assume that the utility of aggregate good consumption and the utility of labor supply take the following forms: $U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma}$, and $V(N_{jt}) = \frac{N_{jt}^{1+\varphi}}{1-\varphi}$, where σ is the inverse of the inter-temporal elasticity of substitution and the parameter φ is the inverse of the Frisch elasticity of labor supply. The inter-temporal consumption decision leads to the following Euler equation:

$$C_t^{-\sigma} = \beta(1+i_t)E_t^H \left(C_{t+1}^{-\sigma} \frac{P_t}{P_{t+1}} \right). \tag{5}$$

Equation (5) shows that consumption decisions are forward-looking. Current demand depends the relative cost of consumption today versus consumption tomorrow.

The intra-temporal labor supply decision sets the marginal rate of substitution between leisure

and consumption equal to the real wage:

$$\frac{N_t^{\varphi}(j)}{C_t^{-\sigma}} = \frac{W_t}{P_t}. \tag{6}$$

2.2.2 Firms

Firms make two decisions to maximize expected profits: the intra-period cost minimization and the optimal pricing decisions. As the cost minimization problem only involves information within the island and information is symmetric within islands, the intra-period cost minimization problem is free from informational frictions. The optimal pricing decision, by contrast, is affected by both the Calvo price rigidity and the informational frictions. In each period, a measure $1-\theta$ of firms get the Calvo lottery to reset their prices. Other firms charge their previous prices. A firm j that resets its price in period t chooses $P_t^*(j)$ to maximize its own expectation of the sum of all discounted profits while $P_t^*(j)$ remains effective. The profit optimization problem can be written as follows:

$$max_{P_t^*(j)} \Sigma_{k=0}^{\infty} \theta^k E_t^j \left\{ Q_{t,t+k} \left[P_t^*(j) Y_{t+k}(j) - U_{t+k}^w(j) W_{t+k}(j) N_t(j) \right] \right\}, \tag{7}$$

where E_t^j denotes firm j's expectation conditional on its information set, ω_j. $Q_{t,t+k}$ is the stochastic discount factor given by: $Q_{t,t+k} = \beta^k \frac{U'(C_{t+k})}{U'(C_t)} \frac{P_t}{P_{t+k}}$. $U_{t+k}^w(j)$ denotes the wage markup for firm j.

Firms face two constraints. The first is the demand for their products, which results from the household's optimal allocation among intermediate goods. The second constraint is the production technology. Following the tradition of New Keynesian literature, I assume that labor is the only input and each firm produces according to a constant return to scale technology,

$$Y_t(j) = A_t(j) L_t(j), \tag{8}$$

where $A_t(j)$ denotes the technology of firm j.

There are two sources of uncertainty that affect the pricing decisions of each firm: technology shocks and wage markup shocks. I assume that both shocks have an aggregate component and an idiosyncratic component. The idiosyncratic components are drawn independently in every period, and are distributed log-normally around their aggregate components.

$$log(A_t(j)) \equiv a_t(j) = a_t + s_t^a(j), \qquad s_t^a(j) \sim N(0,\ \sigma_{sa}^2)$$
$$log(U_t^w(j)) \equiv u_t^w(j) = u_t^w + s_t^u(j), \qquad s_t^u(j) \sim N(0,\ \sigma_{su}^2)$$

I assume that the aggregate components of both shocks follow AR(1) processes:

$$a_t = \phi^a a_{t-1} + v_t^a, \qquad v_t^a \sim N(0,\ \sigma_{va}^2)$$
$$u_t^w = \phi^u u_{t-1}^w + v_t^{uw}, \qquad v_t^{uw} \sim N(0,\ \sigma_{vuw}^2)$$

The first order condition for labor input implies that the nominal marginal cost of production is $U_t(j)W_t(j)/A_t(j)$. Substituting the marginal cost of production into the optimal pricing decision results in

$$P_t^*(j) = \frac{\varepsilon}{\varepsilon - 1} \frac{E_t^j \Sigma(\beta\theta)^k u'(C_{t+k}) P_{t+k}^{\varepsilon} Y_{t+k} u_{t+k}(j) w_{t+k(j)} A_{t+k}^{-1}(j)}{E_t^j \Sigma(\beta\theta)^k u'(C_{t+k}) P_{t+k}^{\varepsilon-1} Y_{t+k}}. \tag{9}$$

Equation (9) implies that individual resetting prices are forward-looking and strategic complements. The optimal resetting price of firm j increases with the expectation of a higher firm-specific marginal cost of production and a higher aggregate price level in both the current and all future periods.

2.3 Aggregation and Equilibrium in the Private Sector

Equilibrium variables in the private sector are solved in log deviations from steady state values (i.e., $x_t \equiv ln(X_t/X)$), and denoted by lower-case letters. (See Appendix A for details.)

The Output Gap

Following the New Keynesian tradition, I express the output in terms of the output gap, $\hat{y}_t$, which is defined as the difference between y_t and the natural level of output, y_t^n. The natural level of output is defined as the output level under flexible prices and perfect information. In this situation, y_t^n becomes a linear function of a_t $y_t^n = \frac{\varphi+\sigma}{1+\varphi} a_t$, and follows an AR(1) process, $y_t^n = \phi y_{t-1}^n + v_t$, where $\phi = \phi^a$, and $\sigma_v = \frac{\varphi+\sigma}{1+\varphi}\sigma_{va}$.

The output gap is derived as follows:

$$\hat{y}_t \equiv y_t - y_t^n = E_t^H \hat{y}_{t+1} - \frac{1}{\sigma}\left[i_t - \left(\frac{1}{1-\phi} r_t^n - \frac{\phi}{1-\phi} E_t^H r_t^n \right) - E_t^H \pi_{t+1} \right], \tag{10}$$

where $E_t^H \hat{y}_{t+1} = E_t^H y_{t+1} - E_t^H y_{t+1}^n = E_t^H y_{t+1} - \phi E_t^H y_t^n$. r_t^n denotes the natural rate of interest, which is the equilibrium real interest rate that equates output to its natural level. It is calculated to be: $r_t^n \equiv \sigma\left(E_t y_{t+1} - y_t^n\right) = \sigma(\phi - 1) y_t^n$.[2]

If information is perfect, $E_t^H r_t^n = r_t^n$, and expectations about future equilibrium are objective i.

[2]The natural rate shock is mapped from the aggregate component in firm technology shocks in the present model, but it can also be other types of demand shocks as well, for example time preference shocks or government spending shocks. As long as the output target in the next period is not known for the household, the expected natural rate affect the output gap in addition to the actual one.

e., $E_t^H \hat{y}_{t+1} = E_t \hat{y}_{t+1}$ and $E_t^H \pi_{t+1} = E_t \pi_{t+1}$. In this case, the IS curve becomes:

$$\hat{y}_t = E_t \hat{y}_t - \frac{1}{\sigma}\left[i_t - r_t^n - E_t \pi_{t+1}\right] \tag{11}$$

Comparing equation (10) and equation (11), we find that the informational frictions enlarge the output gap caused by a natural rate shock under perfect information. As consumption decisions are forward-looking, not knowing the change in future output makes private agents do not adjust current consumption accordingly. The nominal frictions and the informational frictions work in the same direction that drive the equilibrium output away from its efficient level.

Inflation

According to the assumption of Calvo-type price rigidity, the current aggregate price level is the composite of the aggregate price in the previous period and the average resetting prices:

$$p_t = \theta p_{t-1} + (1-\theta)\int p_t^*(j)dj. \tag{12}$$

The integral of resetting prices may potentially lead to the higher order beliefs problem. As equation (9) shows, $p_t^*(j)$ includes firm j's expectation about the aggregate price level P_t, and, thus, includes other firms' expectations. This leads to the infinite regress problem: each firm uses its firm-specific shock as a private signal, and guesses the private signals observed by other firms. I abstract from this higher order beliefs problem by modeling homogeneous, subjective beliefs,[3] which I denote as E_t^s.[4] Homogeneous beliefs are formed when private agents, including both the household and all firms, use only public signals when forming expectations on aggregate variables. Mathematically, this assumption implies that the idiosyncratic components of firm-specific shocks have infinite variance. In this case, private signals are completely uninformative, so that firms do not use their private signals about firm-specific shocks to form beliefs about aggregate variables.[5]

The aggregation of individual resetting prices leads to the New Keynesian Phillips curve under subjective beliefs: (see Appendix A for the detailed derivation.)

$$\pi_t = \beta\theta E_t^s \pi_{t+1} + (1-\theta)E_t^s \pi_t + \kappa\theta\hat{y}_t + u_t, \tag{13}$$

[3]Note that subjective expectations in this paper refer to the rational expectations formed as a result of imperfect information about the state variables.

[4]There are many papers that address how higher order beliefs lead to monetary policy to have more persistent effects, for example Woodford (2001) and Angeletos and La'O (2009). For the solution method to the infinite regress problem, see Huo and Takayama (2015), Melosi (2016) and Nimark (2017).

[5]Another way to generate homogeneous beliefs is to assume that firms have the same technology and face the same wage markup but do not observe them when setting prices. This assumption, however, implies that aggregate inflation consists of only the firms' expectations, and does not consist of actual shocks. Consequently, there will be no trade-off between inflation and the output gap due to the lack of actual cost-push shocks, which makes the optimal monetary policy becomes less interesting.

where $\kappa = \frac{(1-\beta\theta)(1-\theta)(\varphi+\sigma)}{\theta}$, and u_t denotes the cost push shock, which is related to the wage markup shock as $u_t = (1-\theta)(1-\beta\theta)u_t^w$.

If information is perfect, expected inflation is the same as actual inflation, i.e., $E_t^s \pi_t = \pi_t$, and expectations about future equilibrium are objective i.e., $E_t^s \pi_{t+1} = E_t \pi_{t+1}$. In this case, the Phillips curve becomes:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa \hat{y}_t + \frac{1}{\theta} u_t \tag{14}$$

Comparing equation (13) and equation (14), we find that informational frictions reduce inflation fluctuations caused by cost-push shocks. To understand this, recall that under perfect information, a cost-push shock increases aggregate price level for two reasons. First, firms increases their prices due to the increase in their own cost of production. Second, as optimal prices are strategic complements, firms also increase prices when expecting the aggregate price level to go up. Under imperfect information, firms are able to observe their own firm-specific shocks, but in absence of public signals, they do not update beliefs on the aggregate price level. Therefore, firms change their prices by less under imperfect information.

The equilibrium of aggregate variables are summarized by the expression of the output gap and inflation. The underlying aggregate shocks in the private sector are now written in terms of natural rate shocks in the output gap and cost-push shocks in inflation. The process of shocks are given by

$$r_t^n = \phi r_{t-1}^n + v_t,$$
$$u_t = \phi_u u_{t-1} + v_t^u,$$

where the natural-rate shock and the cost-push shock are mapped from the technology shock and the wage markup shock as $r_t^n = \frac{\varphi+\sigma}{1+\varphi}\sigma(\phi-1)a_t$, and $u_t = (1-\theta)(1-\beta\theta)u_t^w$. Denote the auto-coefficients of the natural-rate shock and the cost-push shock as ϕ and ϕ_u. By construction, they are the same as the auto-coefficients of the aggregate technology process and the wage markup process. Denote the standard deviation of the natural-rate shock and the cost-push shock as σ_r and σ_u. By construction, $\sigma_r = \frac{\varphi+\sigma}{1+\varphi}\sigma(\phi-1)\sigma_{va}$, and $\sigma_u = (1-\theta)(1-\beta\theta)\sigma_{vuw}$.

3 Monetary Policy with Serially Uncorrelated Shocks

In this section, I focus on the within-period gains from commitment due to the informational effect of monetary policy. I add two assumptions to make the model static in nature. First, I assume that underlying shocks have no serial correlation. Second, I impose the restriction that the central bank can only respond to current states, which excludes the traditionally studied gains from committing to a delayed response. Under these two assumptions, although private agents are

forward-looking, the expectations of future equilibrium variables do not affect current decisions, as future equilibrium variables are expected to be at their steady state levels.

A discretionary central bank observes the realized shocks before making interest rate decisions. It optimizes its interest rate decision at every state, taking as given how beliefs are formed in the private sector. Under rational expectations, the private agents' beliefs on the interest rate response function are the same as the central bank's best response in equilibrium. In comparison, a central bank with commitment chooses a state-contingent policy rule prior to the realization of shocks. If the central bank credibly commits to a rule that is different from the equilibrium response under discretion, it is able to change the informational effect of the interest rate. The differences between discretionary monetary policy and policy commitment are summarized in the following event flow.

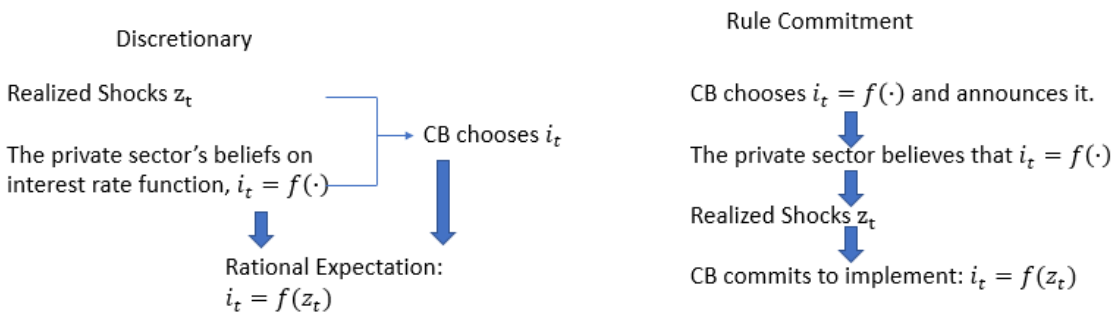

Figure 1: Discretionary Monetary Policy versus State-Contingent Policy Rule

In this section, I first show how the informational effect of the interest rate is determined by the interest rate response function. Then, I analyze how committing to a rule can change the informational effect, which consequently leads to the gains from commitment.

3.1 Equilibrium in the Private Sector under an Arbitrary Interest Rate Policy

In this section, I characterize the equilibrium output gap and inflation under an arbitrary interest rate respond function. I illustrate the two effects that the interest rate has on equilibrium inflation and the output gap: the traditionally studied direct effect on the borrowing cost of the household and the informational effect on the beliefs in the private sector, which is determined by the interest rate response function.

The assumptions of no serial correlation in shocks and no policy commitment to future responses simplify the expressions of the IS curve and the Phillips curve, as future equilibrium is

expected to be at steady state. The output gap and inflation are: [6]

$$\hat{y}_t = -\frac{1}{\sigma}(i_t - r_t^n) \tag{15}$$

$$\pi_t = (1-\theta)E_t^s \pi_t + \kappa\theta\hat{y}_t + u_t. \tag{16}$$

First notice that the output gap is free from expected shocks, which makes the informational effect of interest rates do not play a role. Intuitively, the household's inter-temporal optimization is not affected by subjective beliefs when current prices are observable and future equilibrium variables are expected to be at steady state levels. In contrast, inflation is affected by subjective beliefs, as individual firms do not observe the aggregate price level when setting prices. Substituting expected inflation and the expected output gap leads to the expression of inflation in terms of the output gap, the actual shocks and the expected shocks:

$$\pi_t = \kappa\hat{y}_t + (1-\theta)\frac{\kappa}{\sigma}(E_t^s r_t^n - r_t^n) + \frac{1-\theta}{\theta}E_t^s u_t + u_t. \tag{17}$$

Equation (15) and equation (17) show the two effects that the interest rate have on the equilibrium in the private sector: the first one is the direct effect, which is the conventionally studied effect on the borrowing cost of the household. Through the direct effect, an increase in the interest rate reduces current consumption, as it increases the relative cost of current consumption versus future consumption. At the same time, the direct effect of an increase in the interest rate also reduces the aggregate price level, as each firm reduces its resetting price due to a lower demand. The direct effect of the interest rate on the output gap and inflation are summarized as follows:

$$\frac{\partial \hat{y}_t}{\partial i_t}|_{direct} = -\frac{1}{\sigma}, \tag{18}$$

$$\frac{\partial \pi_t}{\partial i_t}|_{direct} = \frac{\partial \pi_t}{\partial \hat{y}_t}\frac{\partial \hat{y}_t}{\partial i_t} = -\frac{\kappa}{\sigma}. \tag{19}$$

There is no informational effect of the interest rate on the output gap, and the informational effect of the interest rate on inflation is the combination of the changes in expected natural rate shock and the expected cost push shock. Inflation increases with expected cost-push shock, because each firm increases its price when expecting the aggregate price level to go up due to the aggregate wage mark-up shock. Expected natural-rate shock also increases inflation, because each firm increases

[6] Following the conventional New Keynesian literature, the long-run distortion has been eliminated via Pigouvian tax as an employment subsidy, so that the steady state levels of the output gap and inflation are all zero.

its price when expecting a higher aggregate demand.

$$\frac{\partial \hat{y}_t}{\partial i_t}|_{informational} = 0, \tag{20}$$

$$\frac{\partial \pi_t}{\partial i_t}|_{informational} = \frac{\partial \pi_t}{\partial E_t^s r_t^n}\frac{\partial E_t^s r_t^n}{\partial i_t} + \frac{\partial \pi_t}{\partial E_t^s u_t}\frac{\partial E_t^s u_t}{\partial i_t}. \tag{21}$$

where the partial derivatives of inflation on the expected natural-rate and the expected cost-push shock are defined in equation (17) as: $\frac{\partial E_t^s r_t^n}{\partial i_t} = (1-\theta)\frac{\kappa}{\sigma}$, $\frac{\partial E_t^s u_t}{\partial i_t} = \frac{1-\theta}{\theta}$.

Belief Formation

Consider an arbitrary linear interest rate function which responds linearly to the two aggregate shocks, i.e., $i_t = F_r r_t^n + F_u u_t$. When private agents have rational expectations, the interest rate becomes *one signal* that simultaneously provides information about *two shocks*. When private agents extract information from the interest rate about one shock, the prior distribution of the other shock becomes the source of noise in this signal.

Agents in the private sector are Bayesian, and form best linear forecasts by optimally weighting their prior beliefs (shocks have zero ex-ante mean) and the current signal (the interest rate). Let K_r and K_u denote the optimal weights on the two states after observing interest rate changes, which are determined through the optimal filtering process. Beliefs formed through the Kalman Filtering process are given by,

$$\begin{bmatrix} E_t^s r_t^n \\ E_t^s u_t \end{bmatrix} = \begin{bmatrix} 1-K_r \\ 1-K_u \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} K_r \\ K_u \end{bmatrix}\hat{i}_t = \begin{bmatrix} K_r F_r & K_r F_u \\ K_u F_r & K_u F_u \end{bmatrix}\begin{bmatrix} r_t \\ u_t \end{bmatrix}, \tag{22}$$

where

$$K_r F_r = \frac{F_r^2 \sigma_r^2}{F_r^2 \sigma_r^2 + F_u^2 \sigma_u^2}, \tag{23}$$

$$K_u F_u = \frac{F_u^2 \sigma_u^2}{F_r^2 \sigma_r^2 + F_u^2 \sigma_u^2}. \tag{24}$$

The above system of equations solve the beliefs formation process in the private sector. The following lemma provides an interpretation:

Lemma 1: *Beliefs are more sensitive to the shock (1) to which the interest rate responds more aggressively, and (2) that has higher ex-ante dispersion.*

Lemma 1 describes how the precision of the informational effect of the interest rate is determined by the interest rate response function and the ex-ante dispersion of the underlying shocks. When private agents observe a change in the interest rate, they cannot distinguish whether the change is a response to the natural-rate shock or the cost-push shock. When the two shocks have

same ex-ante dispersion ($\sigma_r = \sigma_u$), private agents believe that the interest rate is more likely to respond to the shock to which it is more sensitive. For example, if the interest rate barely responds to cost-push shocks, then after observing a change in the interest rate, agents in the private sector infer that the change in the interest rate is less likely to be a response to a cost-push shock. Otherwise, provided that F_u is very small, the change in the interest rate has to come from a large cost-push shock, which is less likely to realize given the fact that shocks have same prior distribution. When the interest rate is equally sensitive to both shocks, ($F_r = F_u$) private agents in the private sector update more toward the shock that has higher ex-ante dispersion, as the ex-ante mean of the shock has a smaller weight in the belief-formation process.

3.2 Discretionary Monetary Policy

This section analyzes the equilibrium interest rate set by an optimizing, discretionary central bank which takes as given the informational effect of its policy decisions. The private sector has rational expectations on the equilibrium response function of the interest rate, from which it extracts information on the underlying shocks. Simultaneously, the household makes consumption decisions and firms make pricing decisions.

3.2.1 The Phillips Curve

The Phillips curve describes the constraint faced by the discretionary central bank, as it captures the trade-off between output gap stabilization and inflation stabilization. Recall that with perfect information, the slope of the Phillips curve is exogenous to the interest rate decision. Moreover, the Phillips curve crosses the origin of the $(\hat{y}_t,\ \pi_t)$ plane after a natural-rate shock and has a positive intercept after a positive cost-push shock.

In contrast, with imperfect information, the Phillips curve depends not only on the realization of actual shocks, but also on the expectations about the shocks, which further depends on the expectations on the interest rate reaction function. When private agents expect that the interest rate in equilibrium reacts positively to both shocks in a linear way, which is given by $i_t = F_r r_t^n + F_u u_t$. the Phillips curve becomes:

$$\pi_t = \left\{ \kappa - \sigma \left[(1-\theta)\frac{\kappa}{\sigma} K_r + \frac{1-\theta}{\theta} K_u \right] \right\} \hat{y}_t + \left\{ (1-\theta)\frac{\kappa}{\sigma}(K_r - 1) + \frac{1-\theta}{\theta} K_u \right\} r_t^n + u, \qquad (25)$$

where K_r and K_u are determined through the optimal filtering process in equation (22). The following figure plots the Phillips curve under imperfect information, in comparison with the Phillips curve under perfect information.[7]

[7] see Section 6 or parameter values

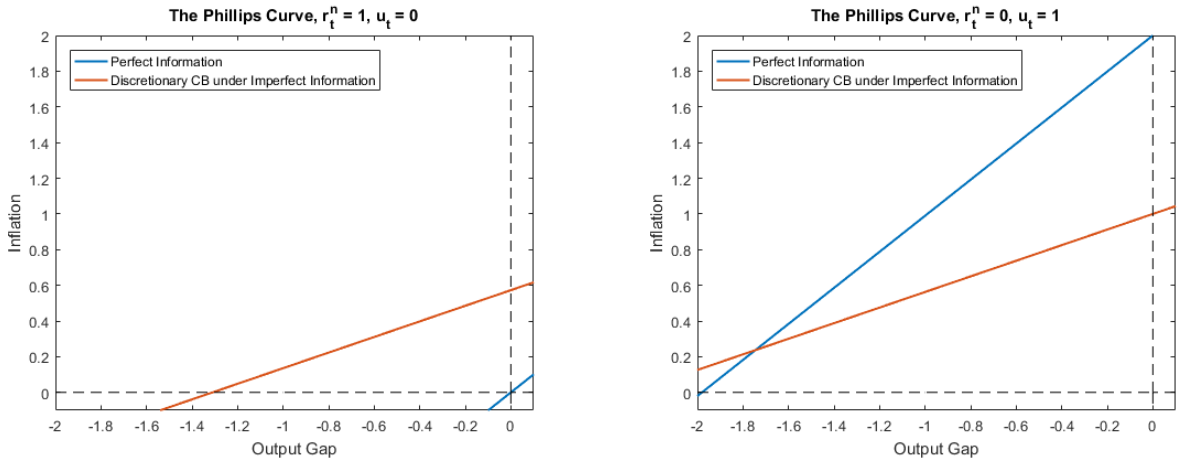

Figure 2: The Phillips Curve under Discretionary Monetary Policy

The above figures plot the Phillips curve after a natural-rate shock (left) and a cost-push shock (right) for a discretionary central bank which is expected to play the equilibrium interest rate. The prior distribution of the two shocks are set equal, $\sigma_r = \sigma_u = 0.3$ See Section 5 for values of other parameters.

The following lemma summarizes the differences between the Phillips curve under perfect information and the Phillips curve under imperfect information.

Lemma 2: *With private agents expect that in equilibrium, the interest rate responds to both shocks positively, the informational effect of the interest rate changes the Phillips curve in three aspects, relative to the Phillips curve under perfect information:*

1. The slope of the Phillips curve is flatter than that under perfect information.

2. The intercept after a cost push shock is reduced.

3. There is non-zero intercept after a natural rate shock.

The reduction in the positive slope of the Phillips curve captures the fact that the informational effect dampens the direct effect of monetary policy. When interest rate increases, its direct effect increases the cost of borrowing of household, which reduces both the output gap and inflation, as shown in equation (18 - 19). At the same time, the informational effect leads to positive updates on both of the two shocks, making firms increase prices as they expected a higher aggregate demand and a higher aggregate price level, as shown in equation (20 - 21).

The intercept of the Phillips curve captures the situation when the interest rate responds to close the output gap. After a cost-push shock, closing the output gap means interest rate does not respond. Absence of any signals, private agents do not update beliefs on both shocks, and firms increase prices only due to the increases in their own cost of production. Consequently, inflation is lower under imperfect information.

After a natural-rate shock, closing the output gap requires the interest rate responds one-to-one to the realized natural-rate shock. The direct effect of such tightening monetary policy reduces

inflation, but at the same time, private agents do not know whether the interest rate is responding to a natural-rate shock or a cost-push shock. There are two forces that govern the sign of the intercept of the Phillips curve. First, the expected output gap is lower than the actual output gap, as the expected natural-rate shock is lower than the actual natural-rate shock. This reduces the intercept, as firms reduce prices when expecting a lower demand than the actual one. Second, as firms positively update their expectations on the cost-push shock, they increase prices as they expect a higher aggregate price level.

The changes in the Phillips curve imply the costs and benefits of informational frictions. On the good side, inefficient coordination on the cost-push shock reduces, as there is less information about the shock. On the bad side, the effectiveness of monetary policy is reduced, as the informational effect of interest rate dampens its direct effect to stabilize the economy.

3.2.2 Optimal Discretionary Monetary Policy

The objective function of a central bank is to minimize the sum of squared output gap and squared inflation for all periods. Due to the static nature of this section, the objective function of the discretionary central bank reduces to minimizing current deviations, which is given by:

$$min_{i_t} L(t) = \begin{bmatrix} \pi_t & \hat{y}_t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} \pi_t \\ \hat{y}_t \end{bmatrix} + indept.\, terms \tag{26}$$

subject to

$$\hat{y}_t = -\frac{1}{\sigma}(i_t - r_t^n) \tag{27}$$

$$\pi_t = \kappa \hat{y}_t + (1-\theta)\frac{\kappa}{\sigma}(E_t^s r_t^n - r_t^n) + \frac{1-\theta}{\theta} E_t^s u_t + u_t \tag{28}$$

$$E_t^s r_t^n = K_r i_t \tag{29}$$

$$E_t^s u_t = K_u i_t \tag{30}$$

where ω is a constant that results from the second-order approximation of the household's utility.[8]

Definition: *A Markov perfect equilibrium between a discretionary central bank and the private sector with rational expectations can be described in aggregate terms in the following way:*

(i) Inflation and the output gap result from the household's optimal consumption choices and firms' optimal price-setting behaviors, which are shown in equations (10) and (13).

[8] see Woodford (2011) for general derivation of the second-order approximation of the household's utility under perfect information, and Adam (2007) for the application to imperfect information. Appendix C shows the derivation that applies to the specific assumptions in this paper.

(ii) Beliefs in the private sector about the realization of shocks are formed through the Kalman Filtering process as shown in equation (22 - 24);

(iii) The interest rate is set by the central bank's optimization problem as specified in (26), subject to the constraints as specified in equation (27 - 30).

The solution of the optimizing interest rate involves a circularity problem, as the constraint that the central bank faces depends on the optimal decision of the interest rate under rational expectations. I follow the method of Svensson and Woodford (2003) to find the solution of the optimizing interest rate. First, I conjecture an interest rate reaction function, $i_t = F_r^0 r_t^n + F_u^0 u_t$, which determines the Phillips curve. Then, the central bank chooses the interest rate to maximize its objective function under the constraint of the Phillips curve. The equilibrium interest rate under rational expectations is found as the fixed point between the conjectured interest rate function and the optimizing interest rate solution. The following lemma characterizes the optimizing discretionary interest rate.[9]

Lemma 3: *When shocks are not serially correlated, the discretionary central bank targets a negative relation between inflation and the output gap, and the absolute value of the targeted ratio between inflation and the output gap is higher under imperfect information than under perfect information.*

The intuition of this result is as follows: the discretionary optimization problem yields that the optimal combination of inflation and the output gap is the tangent point between the indifference curve of the central bank's loss function and the Phillips curve. In addition, the slope of the Phillips curve is reduced by the informational effect of the interest rate, which makes the vector of $(\hat{y}_t^*,\ \pi_t^*)$ steeper in equilibrium.

Precisely speaking, the targeted ratio between inflation and output gap results from the first order condition of the central bank's optimization problem, which is given by:

$$\pi_t = -\left(\frac{\partial \pi_t}{\partial i_t^*}\right)^{-1} \frac{\partial \hat{y}_t}{\partial i_t^*} \omega \hat{y}_t \equiv= -R\hat{y}_t. \tag{31}$$

Under full information, the absolute value of the targeted ratio between inflation and the output gap is given by:

$$R_{perfect\ info} = \left(\frac{\partial \pi_t}{\partial i_t^*}\right)^{-1} \frac{\partial \hat{y}_t}{\partial i_t^*} \omega = \left(-\frac{\kappa}{\sigma}\right)^{-1} \left(-\frac{1}{\sigma}\right) \omega. \tag{32}$$

With informational frictions, the marginal effect of the interest rate on inflation is dampened

[9] A detailed derivation for solving for the equilibrium optimizing interest rate is provided in Appendix B.

by the informational effect, and thus

$$R_{imperfect\ info} = \left(\frac{\partial \pi_t}{\partial i_t^*}\right)^{-1} \frac{\partial \hat{y}_t}{\partial i_t^*} \omega = \left(-\frac{\kappa}{\sigma} + (1-\theta)\frac{\kappa}{\sigma}K_r + \frac{1-\theta}{\theta}K_u\right)^{-1} \left(-\frac{1}{\sigma}\right) \omega. \tag{33}$$

Since monetary policy should always counteract the effect of shocks, the optimizing interest rate responds positively to both shocks, i.e., $F_r > 0$, $F_u > 0$. Thus, the the Kalman gains are both positive, i.e., $K_r > 0$, $K_u > 0$, which yield that $R_{imperfect\ info} > R_{perfect\ info}$.

3.3 Monetary Policy Rule

A central bank with credible commitment chooses a state-contingent policy rule prior to the realization of shocks. After the realization of shocks, the central bank perfectly observes the shocks and implements the interest rate implied by the rule. Private agents observe the interest rate, form beliefs about the realized shocks and simultaneously choose consumption and pricing decisions. As stated in Lemma 1, under rational expectations, how the interest rate reacts to different shocks determines its informational effect. Therefore, the Phillips curve, which is a function of both the actual and the expected shocks, becomes endogenous to the optimal policy rule decision.

Previous literature has studied the situation in which committing to a path of future interest rates changes the Phillips curve by changing expectations on future equilibrium. To isolate the novel, informational gains from commitment, I restricts that the policy rule only responds to current shocks, and shows the within-period informational gains from commitment.

3.3.1 The Phillips Curve

To understand how committing to a policy rule changes the Phillips curve, let us first consider the following example: suppose the central bank commits to the interest rate rule that tracks the natural rate one-to-one, and does not respond to the cost-push shock, i.e., $(F_r^c,\ F_u^c) = (1,\ 0)$. In this situation, the interest rate becomes a perfect signal about the natural rate shock, which makes $K_r^c = 1$ and $K_u^c = 0$. Consequently, the intercept after a natural rate shock becomes zero. The following figure plots the Phillips curve under this commitment, in comparison with the equilibrium under discretionary optimization.

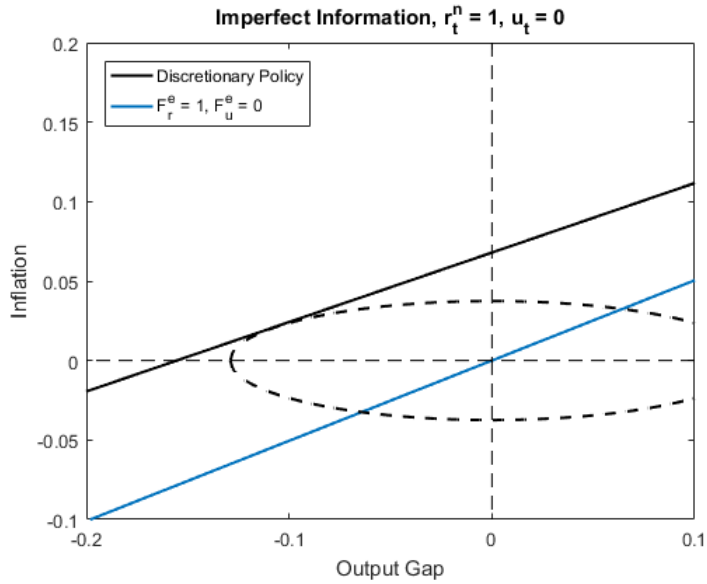

Figure 3: The Phillips Curve under the Rule $i_t = r_t^n$ and under Discretionary Optimization

The black line is the Phillips curve for the discretionary central bank which is expected to play the equilibrium interest rate. The blue line is the Phillips curve for the central bank which is expected to play the rule such that $i_t = r_t^n$. The dotted ellipse denotes the indifference curve of the central bank's loss function, $L = \pi_t^2 + \omega \hat{y}_t^2$.

Under such commitment, although the slope of the Phillips curve is still flatter than the one under perfect information, the Phillips curve now crosses the origin of the $(\hat{y}_t,\ \pi_t)$ plane, making dual stabilization available.

There are three properties worth mentioning. First, dual stabilization is not only determined by the actual shock and the response of the interest rate to this shock, as the case under perfect information, but it is also determined by the expectations on how the interest rate would react if a cost push shock is realized. If private agents expect that the interest rate will react positively to a cost-push shock as well, the interest rate cannot be a perfect signal of the natural-rate shock.

Second, the informational effect of the interest rate, which is determined by how private agents expect the interest rate to respond to different shocks, shifts the Phillips curve. Tracing along the Phillips curve captures the direct effect of the interest rate. As the Phillips curve curve is now endogenous to the policy rule decision, there is only one point along the Phillips curve that is consistent with such commitment under rational expectations, which is the origin of the plane, i.e., $(\hat{y}_t,\ \pi_t) = (0,0)$. Other points captures the equilibrium when private agents have been convinced that by the rule, so the informational effect of the interest rate is fixed, but the central bank deviates from this rule. For example, the Phillips curve below $\pi_t = 0$ is the equilibrium in which the interest rate acts stronger than tracking one-to-one with the natural-rate shock.

Third, although this policy rule achieves the first-best after a natural rate shock, it is not the

optimal policy rule. This is because the central bank also cares about the equilibrium after a cost-push shock, in which case being completely inelastic to a cost-push shock is not optimal. The optimal policy rule is solved in the next section.

3.3.2 Optimal Policy Rule

The central bank with commitment chooses the interest rate feedback rule $i_t = f(r_t^n,\ u_t,\ \pi_t,\ \hat{y}_t)$ prior to the realization of shocks, which becomes $i_t = F_r^c r_t^n + F_u^c u_t$ in equilibrium. The optimal rule is found by choosing F_r and F_u to minimize the central bank's ex-ante loss over the state space:

$$min_{F_r^c,\ F_u^c} \int\int \pi_t^2(r_t,\ u_t) + \omega \hat{y}_t^2(r_t,\ u_t) dr_t^n du_t, \tag{34}$$

subject to

$$\hat{y}_t = -\frac{1}{\sigma}\left[(F_r^c - 1)r_t^n + F_u^c u_t\right], \tag{35}$$

$$\pi_t = \left\{-\frac{\kappa}{\sigma}(F_r^c - 1) + (1-\theta)\frac{\kappa}{\sigma}(K_r F_r^c - 1) + \frac{1-\theta}{\theta}K_u F_r^c\right\} r_t^n \tag{36}$$

$$+\left\{-\frac{\kappa}{\sigma}F_u^c + (1-\theta)\frac{\kappa}{\sigma}K_r F_u^c + \frac{1-\theta}{\theta}K_u F_u^c + 1\right\} u_t$$

$$E_t^s r_t^n = K_r F_r^c r_t^n + K_r F_u^c u_t, \tag{37}$$

$$E_t^s u_t = K_u F_u^c r_t^n + K_u F_u^c u_t. \tag{38}$$

where $(K_r,\ K_u)$ are functions of $(F_r^c,\ F_u^c)$ as specified in equation (18).

Comparing the optimization problem under commitment and the one under discretion, we find that the ability to commit essentially relaxes one constraint: under discretion, private agents expect the central bank to react as its best response after each shock, i.e., $(F_r^e, F_u^e) = \left(F_r^d, F_u^d\right)$, which determines the sensitivity of expected shocks to changes in the interest rate, as measured in $\left(K_r^d,\ K_u^d\right)$. The central bank cannot change the expectations on the interest-rate behavior, as private agents believe the central bank will optimize at all states. In comparison, the central bank with credible commitment can announces that it will implement a sub-optimal interest rate in some states, i.e., $(F_r^c,\ F_u^c) \neq \left(F_r^d,\ F_u^d\right)$. Under rational expectations, the change in expectations on the interest-rate behaviors changes the sensitivity of expected shocks to interest-rate changes. In other words, the central bank with commitment can choose a direct mapping from actual shocks to expected shocks, as it controls the informational effect of the interest rate.

We now turn to comparing the optimal policy rule with the equilibrium interest rate under

discretion. The first-order condition on F_r after the r_t^n shock is

$$\underbrace{-\frac{\kappa}{\sigma}}_{direct}+\underbrace{\Omega_r K_r+\Omega_u K_u}_{informational}+\underbrace{\Omega_r\frac{\partial K_r}{\partial F_r}F_r+\Omega_u\frac{\partial K_u}{\partial F_r}F_r}_{change\ of\ informational\ effect}=\frac{\omega}{\sigma}\frac{\hat{y}_t}{\pi_t}, \tag{39}$$

where $\Omega_r=(1-\theta)\frac{\kappa}{\sigma}$, and $\Omega_u=\frac{1-\theta}{\theta}K_u$. Similarly, the first-order condition on F_u after the u_t shock is

$$\underbrace{-\frac{\kappa}{\sigma}}_{direct}+\underbrace{\Omega_r K_r+\Omega_u K_u}_{informational}+\underbrace{\Omega_r\frac{\partial K_r}{\partial F_u}F_u+\Omega_u\frac{\partial K_u}{\partial F_u}F_u}_{change\ of\ informational\ effect}=\frac{\omega}{\sigma}\frac{\hat{y}_t}{\pi_t} \tag{40}$$

In comparison, rewrite the first-order condition for the discretionary central bank in the same form yields:

$$\underbrace{-\frac{\kappa}{\sigma}}_{direct}+\underbrace{\Omega_r K_r+\Omega_u K_u}_{informational}=\frac{\omega}{\sigma}\frac{\hat{y}_t}{\pi_t} \tag{41}$$

The differences between equation (39), (40) and equation (41) show that the central bank with commitment internalizes the change of the informational effect of interest rate, whereas the discretionary central bank takes the informational effect as exogenous.

Due to the model specifications, the ex-ante welfare under commitment is always weakly better than that under discretion, as the choice set of optimization under commitment includes the equilibrium response under discretion. The question is then whether the gains are strictly positive, and the answer is yes as long as the optimal policy is found to be different from the equilibrium interest rate under discretion.

The gains from commitment can be analyzed through the lens of the changes in the Phillips curve. The central bank wants to reduce the trade-off between inflation stabilization and output gap stabilization, which translates into 1) steepening the slope and 2) reducing the intercept of the Phillips curve. Equation (25) shows that when the Phillips curve has a positive intercept after a natural-rate shock under discretion, these two targets are aligned.[10] The gains from commitment is summarized in the following proposition.

Proposition 1: *When the Phillips curve under discretionary optimization has a positive slope after a natural-rate shock, optimal policy rule improves ex-ante welfare by reducing the degree to which the informational effect dampens the direct effect of the interest rate.*

Specifically, to reduce the informational effect of the interest rate means to reduce the sum of sensitivities of expected shocks to changes in the interest rate (K_r and K_u), weighted by the sensitivities of inflation to the expected shocks (Ω_r and Ω_u), respectively. With parameter values

[10] It is possible that the Phillips curve has a negative intercept after the natural rate shock, in which case commitment can still improves ex-ante welfare, but whether it increases or decreases the informational effect is ambiguous.

taken from traditional literature,[11] the optimal policy rule and the equilibrium interest rate under discretion are found as follows:

$$i_t^c = 1.2808 r_t^n + 1.5265 u_t \tag{42}$$

$$i_t^d = 1.1390 r_t^n + 1.3951 u_t \tag{43}$$

First,the ratio of F_r over F_u is greater under optimal policy rule. As indicated by Lemma 1, optimal policy rule reduces the sensitivity of expected cost-push shocks to actual cost-push shocks ($K_u \cdot F_u$). Second, both F_r and F_u are higher under optimal policy rule. For both shocks, the sensitivity of expected shocks to interest rate changes (K) is smaller for a given sensitivity of expected shocks to actual shocks ($K \cdot F$). Intuitively, for a given change of the interest rate, the private sector believe the realized shock is of a smaller size when they know the interest rate is very sensitive.

Both of the two changes make K_u smaller under commitment, whereas the relative size of K_r is ambiguous. The intuition is as follows. First, the central bank wants to minimize the belief updates on cost-push shocks, no matter the actual shock is a natural-rate shock or a cost-push shock. This is because expectations about a cost-push shock lead to expectations about inflation bias, as private agents expect the central bank to tolerate positive inflation. By contrast, the central bank wants private agents to update beliefs on the natural-rate shock only after an actual natural-rate shock is realized. This is because under perfect information, the central bank is able to achieve dual stabilization after natural-rate shocks. However, as expected natural-rate shock becomes sensitive to interest-rate changes, private agents also mistakenly update expected natural-rate shock when actually the interest rate is responding to a cost-push shock.

3.4 Time Inconsistency Problem

Associated with the gains from commitment is the time inconsistency problem. Policy commitment changes the private-sector expectations of underlying shocks, which shifts the Phillips curve. However once expectations in the private sector are arranged in the desirable way, the Phillips curve is fixed in the eyes of a discretionary central bank. It then wants to re-optimize its interest rate decisions, as long as the deviation is not anticipated by private agents and thus the informational effect of the interest rate remains unchanged.

Figure 4 illustrates both the gains from commitment and the incentives of deviation by comparing the equilibrium output gap and inflation under (1) discretionary optimization, (2) optimal policy rule and (3) the rule such that $(F_r, F_u) = (1, 0)$, respectively.

[11] See Section 6 for parameter values.

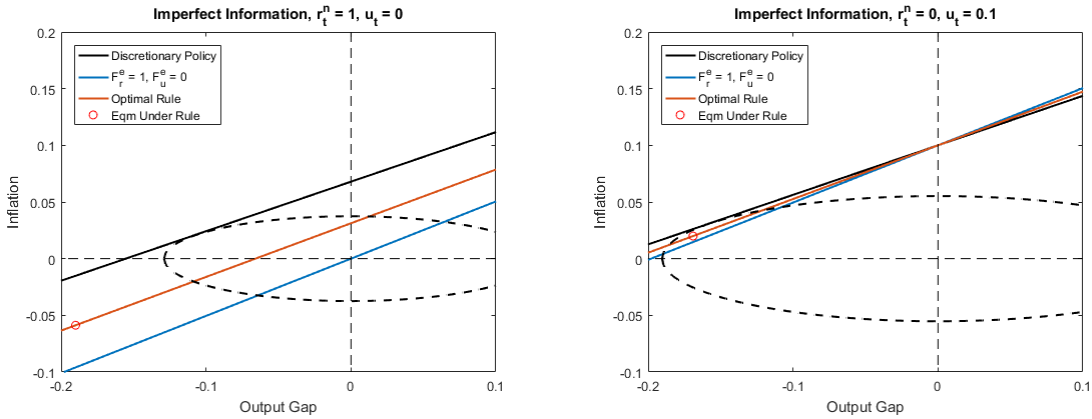

Figure 4: The Gains from Commitment and the Problem of Time Inconsistency

The dotted ellipse is the indifference curve for the central bank whose objective function consists the weighed sum of squared inflation and the squared output gap. The black line is the Phillips curve when the central bank is expected to be discretionary. The red and blue lines are the Phillips curves when the central bank is expected to follow (1) the optimal policy rule and (2) the rule such that $(F_r, F_u) = (1, 0)$, respectively. The red and blue circles are the equilibrium consistent with such commitment.

There are three points worth mentioning. First, after either shocks, the Phillips curve under commitment crosses the indifference curve at which the discretionary central bank achieves optimal equilibrium. Specifically, in both graphs, the slope of the Phillips curve is steeper under commitment. The intercept after the cost-push shock remains unchanged and the intercept after the natural-rate shock is reduced. This shows how reducing the degree to which the informational effect dampens the direct effect of monetary policy can potentially improve the inflation versus output gap trade-off from the equilibrium under discretion.

Second, as pointed out in the previous section, the assumption on rational expectations requires that there is only one point on the Phillips curve under optimal policy rule that is consistent with the commitment. Such point is denoted in red circle. Notice that after both shocks, the equilibrium under optimal policy rule is not the one that is tangent to any indifference curves, meaning the central bank *commits to be sub-optimal* after both shocks. After a natural rate shock, if the central bank decides deviate from the rule, it will reduce the interest rate, hoping to increase inflation and the output gap at the same time. By doing so, the central bank hopes to take the informational advantage which shifts the Phillips curve, without actually sacrificing a lower output gap when a natural-rate shock is realized. However, if private agents anticipate such deviation, they will change their expectations accordingly, making the Phillips curve shift back to the one under discretion.

Third, optimal policy rule does not achieve better outcome in all states. Rather, the central bank with optimal commitment sacrifices the outcome after natural-rate shocks to achieve gains after cost-push shocks. The central bank balances outcomes across all states and achieves welfare

gains ex-ante.[12]

4 External Information

What if the central bank can directly communicate its information to the private sector? The previous literature has discussed the value of central bank communication, and the general consensus is that more precise information about the efficient shocks (natural-rate shocks in this setting) is welfare improving and more precise information about the inefficient shocks (cost-push shocks in this setting) is detrimental.[13] In my model, however, the informational effect of the interest rate complicates the welfare implication of central bank communication, due to the interaction effect between the information revealed through the interest rate and the information conveyed through direct communication.

4.1 Interaction between the Informational Effect of Monetary Policy and Central Bank Direct Communication

To model central bank direct communication, I introduce external public signals about the natural-rate shock and the cost-push shock.[14] Unlike the informational effect through the interest rate, which is restricted by the signal dimension, central bank direct communication is not bounded by the signal dimension.

Denote the external signals sent through the central bank communications as m_t^r and m_t^u, which are distributed log normally around the actual shocks, r_t^n and u_t. The signals received by private agents consist of both the interest rate and external signals sent through the central bank direct communication. Signals are summarized as follows:

$$\begin{bmatrix} \hat{i}_t \\ m_t^r \\ m_t^u \end{bmatrix} = \begin{bmatrix} F_1 & F_3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_t^n \\ u_t \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t^r \\ \varepsilon_t^u \end{bmatrix} \tag{44}$$

The private sector updates beliefs using both the interest rate and the external signals. The

[12]Notice that in the right panel, the unit of cost-push shock is chosen to be 0.1 instead of 1. This is to make comparable to Figure 6 while maintaining the scale of the two figures to be the same.

[13]see Kramer et al. (2008) for survey of literature on central bank communication.

[14]These two shocks can also be interpreted as all other public information apart from the information contained in the interest rate.

belief updating process is given by:

$$\begin{bmatrix} E_t^s r_t^n \\ E_t^s u_t \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \end{bmatrix} \begin{bmatrix} i_t \\ m_t^r \\ m_t^u \end{bmatrix} \tag{45}$$

where the Kalman gains are determined through the optimal filtering process.

As the interest rate is one signal about two shocks, its informational effect interact with the direct communication, which makes the central bank unable to separately convey information about one shock. For a given response function of interest rate, increasing the precision of communication about one shock also increases the precision of information on the other shock. The intuition is that when the central bank communicates more precisely about one shock, the interest rate becomes a less precise signal about that shock, but at at the same time, the interest rate becomes a more precise signal to the other shock. For example, suppose that the central bank provides perfect information about the r_t^n shock. Then suppose that a positive cost-push shock is realized and the natural rate stays at zero. The private agents know that $r_t^n = 0$ from the central bank's perfect communication about the natural rate shock. In addition, the private agents also observe that the interest rate responds positively, and they can also infer perfectly that the increase in the interest rate is due to the positive realization of a cost-push shock.

Figure 5 summarizes the effect on expected shocks when the central bank increases the precision of communication about the natural rate shock, which is modeled by a reduction in the standard deviation of the noise in m_t^r. For both shocks, beliefs are getting closer to the actual shock, i.e., $E_t^s r_t^n \to r_t^n$ after a r_t^n shock, and $E_t^s u_t \to u_t$ after a u_t shock. The next question is whether such change is desirable, which I analyze in the following section.

4.2 Value of (External) Information

Without the informational effect of the interest rate, the welfare implication of information is straight-forward. The central bank wants to reveal full information about the natural-rate shock, because doing so also reveals the fact that it is able to achieve dual stabilization using the direct effect of monetary policy. By contrast, cost-push shock induces a inflation bias under perfect information, making the central bank have to tolerate positive inflation. As private agents have rational expectations, when they expect a positive cost-push shock, firms increase their prices due to higher expected inflation. Consequently, the central bank wants to withhold information about the cost-push shock. In summary, welfare is maximized when the central bank provides perfectly precise signal about the efficient shock (the natural-rate shock), and completely uninformative signal about

the inefficient shock (the cost-push shock).[15]

The existence of the informational effect of the interest rate complicates the welfare effect of central bank communication. As shown in Figure 5, if the central bank provides more precise information about one shock, the private agents can simultaneously have more precise information about the other shock. Figure 6 shows the welfare implication of central bank communication measured by the expected loss for the central bank.

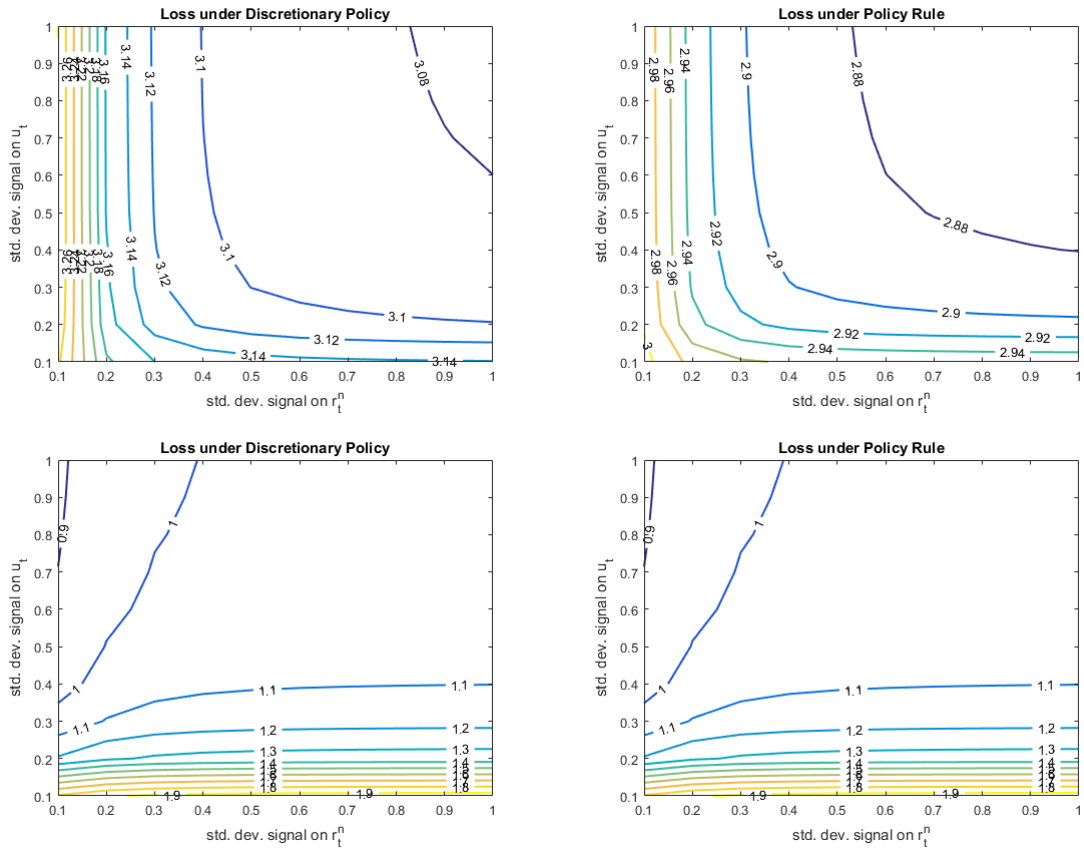

Figure 5: The Value of (External) Information

The first row shows the ex-ante loss of the central bank at varying precision of external signals. In the second row, an implementation error is added to the interest rate, i.e., $i_t = i_t^* + e_t$, with $\sigma_e = 0.5$ and i_t^* being either the equilibrium interest rate under discretion or the optimal policy rule under commitment.

In the first row of Figure 6, I plot the expected loss for the central bank at varying levels of precision of central bank communication under discretion (left) and under commitment (right). It shows that when communication about the cost-push shock becomes more precise, which is modeled by a lower σ_{eu}, the ex-ante loss increases. This is consistent with the conventional wisdom that more precise information about the inefficient shock is welfare reducing. However, when the

[15] For more general discussion on the value of information without the informational effect of monetary policy, see Morris and Shin (2002), Angeletos and Pavan (2007), Angeletos, Iovino and La'O (2016), for examples.

precision of central bank communication about natural-rate shocks increases, the ex-ante loss also increases, which contradicts the conventional wisdom.

There is an implicit assumption. As I assume that there is no implementation error in the interest rate, the interest rate is assumed to be a very precise signal about both shocks. If I relax this assumption by adding an implementation error, the interest rate becomes a less precise signal about both shocks, and the welfare implication of direct communication might change. The second row of Figure 6 plots this case, with a relatively large standard deviation of the implementation error ($\sigma_e = 0.5$). In this situation, the ex-ante loss is minimized when information on natural-rate shock is perfectly precise and information on the cost-push shock is completely zero, which becomes in line with the traditional wisdom.

5 Dynamic Informational Effect

I extend the analysis to incorporate serially correlated shocks, which leads to the dynamic informational effect of interest rates due the persistent learning process in the private sector.

5.1 Equilibrium in the Private Sector

The process of the actual shocks is given by:

$$\begin{bmatrix} r_t^n \\ u_t \end{bmatrix} = \begin{bmatrix} \phi & 0 \\ 0 & \phi^u \end{bmatrix} \begin{bmatrix} r_{t-1}^n \\ u_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_t^r \\ v_t^u \end{bmatrix}. \tag{46}$$

The information set of the private sector includes the values of all parameters and the entire history of interest rates upon t. Due to the serial correlation, private agents optimally weigh current signals and past beliefs (priors) when forming beliefs about current shocks. Due to the persistent belief updating process, current equilibrium is affected past beliefs as well. Therefore, when the interest rate is set to minimize deviations of the current output gap and inflation, it should react to beliefs in the past period as well.

I conjecture that the equilibrium interest rate under discretion is a linear function of all predetermined state variables in period t, which includes both the actual shocks at time t and beliefs in period $t-t$.

$$i_t = F_1 r_t^n + F_2 E_{t-1}^s r_{t-1}^n + F_3 u_t + F_4 E_{t-1}^s u_{t-1}. \tag{47}$$

The private agents have perfect memory of their beliefs in the past. They are able to distinguish the fraction of the interest-rate changes that reacts to current shocks from the fraction of the interest-rate changes that reacts to past beliefs. Let $\hat{i}_t$ denote the fraction of i_t that reacts to current

shocks, which is given by:

$$\hat{i}_t \equiv i_t - F_3 E^s_{t-1} r^n_{t-1} - F_4 E^s_{t-1} u_{t-1} = F_1 r^n_t + F_3 u_t. \tag{48}$$

$\hat{i}_t$ becomes a signal that simultaneously provides information on both shocks.

Beliefs Formation

The private sector forms expectations about current states through the Kalman filtering process. Denote the unobserved state variables as

$$z_t = \Phi z_{t-1} + v_t \tag{49}$$

where $z_t = [r^n_t,\ u_t]'$, $\Phi = \begin{bmatrix} \phi & 0 \\ 0 & \phi^u \end{bmatrix}$, and $v_t = [v^r_t,\ v^u_t]'$ with white noise of variance Q.

Denote the observable signal as

$$s_t = D z_t \tag{50}$$

where $s_t = \hat{i}_t$, and $D = [F_1, F_3]'$.

The Kalman filtering process makes beliefs about the current state variables be the optimal combination of prior beliefs and signals in the current period, which is given by

$$E^s_t z_t = \Phi E^s_{t-1} z^n_{t-1} + K\left(s_t - D\Phi E^s_{t-1} z_{t-1}\right), \tag{51}$$

where the optimal weight, K, is determined by the Ricatti iteration as follows,

$$K = PD'(DPD')^{-1}, \tag{52}$$

$$P = \Phi\left(P - PD'(DPD')^{-1}DP\right)\Phi + Q. \tag{53}$$

Solution in the Private Sector under Arbitrary Policy Coefficients

The equilibrium in the private sector is described by the system of equations summarizing private sector optimization decisions in aggregate variables (equations 10 and 12), shock evolution (equation 46), the interest rate reaction function (equation 47), and belief updating process characterized in equation (48).

To solve the forward-looking variables in equilibrium, I use the method of undetermined coefficients. I first conjecture that $\hat{y}_t$ and π_t are linear functions of the state variables in period t. To economize the use of notations, I use z_t from now on to denote the vector of pre-determined state variables, i.e., $z_t = \left[r^n_t,\ u_t,\ E^s_{t-1} r^n_{t-1},\ E^s_{t-1} u_{t-1}\right]$. The equilibrium output gap and inflation are

given by:

$$\begin{bmatrix} \hat{y}_t \\ \pi_t \end{bmatrix} = \Gamma z_t \tag{54}$$

This conjecture allows for the expression of expected future equilibrium variables in terms of the beliefs about current shocks, $E_t^s r_t^n$ and $E_t^s u_t$:

$$\begin{bmatrix} E_t^s \hat{y}_{t+1} \\ E_t^s \pi_{t+1} \end{bmatrix} = \Gamma E_t^s z_{t+1} \tag{55}$$

where $E_t^s z_{t+1} = [\phi E_t^s r_t^n,\ E_t^s r_t^n,\ \phi^u E_t^s u_t,\ E_t^s u_t]$.

Substituting the expression of the expected future output gap and inflation into the the IS and the Phillips curve results in expressions of $\hat{y}_t$ and π_t as functions of the actual shocks $[r_t^n,\ u_t]$ and the associated expectations, $[E_t^s r_t^n,\ E_t^s u_t]$. Applying the belief-updating process yields the expressions as functions that consist only of predetermined states. (See Appendix B for the detailed derivation.)

5.2 Discretionary Monetary Policy

Due to the persistent learning process, the current interest has a lagged effect. The central bank has *objective expectations* on the lagged effect, as the central bank has perfect information on the realization of shocks and the entire history of beliefs in the private sector. Denote objective expectations by E_t. The information set of the central bank at t includes the entire history of natural-rate and cost-push shocks upon t and the beliefs formed in the private sector upon $t-1$, i.e.,

$$I_t = \left\{ r_T^n,\ E_{T-1}^s r_{T-1}^n\ u_T,\ E_{T-1}^s u_{T-1} \quad \forall T = 0...t \right\}.$$

The central bank's objective expectations on future equilibrium is given by:

$$\begin{bmatrix} E_t \pi_{t+j} \\ E_t \hat{y}_{t+j} \end{bmatrix} = \Gamma E_t z_{t+j}. \tag{56}$$

Note that this is different from equation (48).

The evolution of $E_t z_{t+j}$ includes the auto-correlated actual shocks, and the dynamic process of

belief formation, which can be summarized as

$$\begin{bmatrix} E_t r^n_{t+j} \\ E_t E^s_{t+j-1} r^n_{t+j-1} \\ E_t u_{t+j} \\ E_t E^s_{t+j-1} u_{t+j-1} \end{bmatrix} = \begin{bmatrix} \phi & 0 & 0 & 0 \\ K_{11}F_1 & \phi - K_{11}F_1\phi & K_{11}F_3 & -K_{11}F_3\phi^u \\ 0 & 0 & \phi^u & 0 \\ K_{21}F_1 & -K_{21}F_1\phi & K_{21}F_3 & \phi^u - K_{21}F_3\phi^u \end{bmatrix} \begin{bmatrix} E_t r^n_{t+j-1} \\ E_t E^s_{t+j-2} r^n_{t+j-2} \\ E_t u_{t+j-1} \\ E_t E^s_{t+j-2} u_{t+j-2} \end{bmatrix} \equiv \Lambda E_t z_{t+j-1} \tag{57}$$

where the second row and the forth row come from the Kalman filtering process.

Combining the auto-regressive process of $E_t z_{t+j}$ and the expectations on future equilibrium output gap and inflation as functions of $E_t z_{t+j}$ results in:

$$\begin{bmatrix} E_t \pi_{t+j} \\ E_t \hat{y}_{t+j} \end{bmatrix} = \Gamma \Lambda^{j-1} E_t z_t. \tag{58}$$

The current interest rate affects expectations about realization of current shocks, $E^s_t r^n_t$ and $E^s_t u_t$. Due to the Bayesian learning process, the expectations on current shocks affect expectations on future state variables. Consequently, current interest rate has a lagged effect on future equilibrium. The analysis is summarized in the following lemma.

Lemma 4 *With serially correlated shocks, current interest rates affect future equilibrium through the persistent learning process in the private sector.*

We not turn into how the consideration of the dynamic informational effect changes the equilibrium interest rate under discretion. A discretionary central bank minimizes the expected output gap and inflation deviations in all periods. The central bank's optimization problem can be written as follows:

$$E_t L(t) = E_t[\pi_t^2 + \omega \hat{y}_t^2] + \beta E_t(L(t+1)) \tag{59}$$

where the output gap follows equation (10), inflation follows equation (12), the actual shocks evolve following equation (35), and beliefs are formed using Kalman filtering process specified in equations (37 - 39).

$E_t(L(t+1))$ includes the deviations of equilibrium inflation and the output gap in all future periods:

$$\begin{aligned} E_t\left(L(t_1)\right) &= \Sigma_{j=1}^{\infty} \beta^j E_t \left\{ \begin{bmatrix} \pi_{t+1} & \hat{y}_{t+j} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} \pi_{t+j} \\ \hat{y}_{t+j} \end{bmatrix} \right\} \\ &= \Sigma_{j=1}^{\infty} \beta^j \left\{ \begin{bmatrix} E_t\pi_{t+1} & E_t\hat{y}_{t+j} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} E_t\pi_{t+j} \\ E_t\hat{y}_{t+j} \end{bmatrix} + indept.\, terms \right\} \end{aligned} \tag{60}$$

As the central bank considers the dynamic informational effect of its current interest-rate de-

cisions, the objective function can no longer be reduced to the one-period loss function. The following proposition characterizes the equilibrium interest rate under discretionary optimization.

Proposition 2: *With dynamic informational effect, the optimizing discretionary monetary policy is dynamically "leaning against the wind" as it targets a negative correlation between current and future deviations of the output gap and inflation.*[16] *The consideration of the dynamic informational effect makes the equilibrium interest rate target beliefs in addition to targeting the current inflation and the output gap.*

To understand this proposition, first express the first-order condition of the central bank's objective function:

$$\left\{\frac{\partial E_t\pi_t}{\partial i_t^*}E_t\pi_t+\omega\frac{\partial E_t\hat{y}_t}{\partial i_t^*}E_t\hat{y}_t\right\}=-\frac{1}{2}\Sigma_{j=1}^{\infty}\beta^j\left\{\frac{\partial E_t\pi_{t+j}}{\partial i_t^*}E_t\pi_{t+j}+\omega\frac{\partial E_t\hat{y}_{t+j}}{\partial i_t^*}E_t\hat{y}_{t+j}\right\} \tag{61}$$

As Lemma 5 indicates, the effect of current interest rates on future equilibrium inflation and the output gap comes entirely on its dynamic informational effect. The first order condition can be written as:

$$\left\{\frac{\partial E_t\pi_t}{\partial i_t^*}E_t\pi_t+\omega\frac{\partial E_t\hat{y}_t}{\partial i_t^*}E_t\hat{y}_t\right\}+\frac{1}{2}\Sigma_{j=1}^{\infty}\beta^j\Delta(j-1)=0 \tag{62}$$

where Δ captures how the current interest rate affects future deviations through its informational effect on $[E_t^s r_t^n,\ E_t^s u_t]'$. (See the Appendix for the derivations.)

The consideration of the dynamic informational effect consists of two parts. First, as both consumption and pricing decisions are forward-looking, the discretionary central bank takes into account that as it changes current beliefs, it changes expectations on future equilibrium which in turn changes current equilibrium. The first part is captured in the first two terms in equation (62). Second, the central bank also takes into account that by changing current beliefs, it changes the state variables for future periods, which is in addition to stabilizing the current economy.

Finding the equilibrium interest rate relies on numerical solution. I first conjecture that the interest rate follows a linear function, $i_t = F_1 r_t^n + F_2 E_{t-1}^s r_{t-1}^n + F_3 u_t + F_4 E_{t-1} u_{t-1}$, with which the private sector updates beliefs on $E_t^s r_t^n$ and $E_t^s u_t$. The central bank then chooses the interest rate to minimize its loss function. The equilibrium interest rate is found when there is a fixed-point between the conjectured interest rate and the optimal interest rate decision. Details of this solution method are provided in Appendix B.

The persistence in underlying shocks strengthens the informational effect of interest rate, because it increases the effect of expected future deviations on current consumption and pricing deci-

[16] As long as there are shocks that the central bank is unable to completely offset, optimal policy can be described as "leaning against the wind" - seeking a contemporary negative correlation between the output gap and inflation. For discussion about the conventional within-period "leaning against" policy that is caused by informational frictions, see Adam (2005), Angeletos and La'O (2013), and Tang (2015), among others.

sions. If the serial correlation is high enough, it may cause optimal discretionary interest rate to fail to exist. The intuition is the following. Suppose that the private sector believes the best response of central bank is to increase the interest rate to the two shocks. If cost push shock is realized to be positive, which makes the inflation positively deviate from steady-state, the nominal effect of the interest rate decreases inflation and the informational effect of the interest rate increases inflation. If the informational effect dominates the direct effect, the inflation increases even further. As a result, a discretionary central bank wants to choose a negative interest rate, which contradicts the beliefs in the private sector that the best response of interest rate is to react positively to the two shocks.

5.3 Monetary Policy Rule

As private agents are forward-looking, committing to a path-dependent policy rule is able to change the expectations on future equilibrium, which leads to the traditionally studied gains from commitment. Different from the static model in which I shut down this traditionally studied gains from commitment, I now allow it to interact with the informational gains from commitment.[17]

Potentially, the policy rule can react to current cost-push shocks by a lesser extent and commits to a large response in later periods. In doing so, not only does interest rate reveal less information about the current realization of the cost-push shock, it also decreases expected future inflation through committing to a future tightening policy. The traditional gains from committing to a delayed response strengthen the gains from the informational effect.

The objective function for the committed central bank is the same as the discretionary central bank. In equilibrium, the optimal rule follows same functional form as the discretionary interest rate, i.e., $i_t = F_1 r_t^n + F_2 E_{t-1}^s r_{t-1}^n + F_3 u_t + F_4 E_{t-1}^s u_{t-1}$. The coefficients of the optimal rule, $[F_1,\ F_2,\ F_3,\ F_4]$ are selected to minimize the ex-ante loss from the steady state. [18]

$$min_{F_1,\ F_2,\ F_3,\ F_4} E_t L(t) = \int\int \left(\pi_t^2 + \omega \hat{y}_t^2 + \beta E_t L(t+1)\right) dr_t^n du_t \tag{63}$$

where output gap follows equation (10), inflation follows equation (12), actual shocks evolve as equation (34), and beliefs are formed using Kalman filtering process as specified in equation (39 - 41). Finding the optimal policy rule involves numerical methods. I characterize the optimal policy rule and compare it with the equilibrium interest rate under discretion, in a numerical example in the following section.

[17] I need to make an assumption that current interest rates only react to past beliefs and do not directly react to past actual shocks. Otherwise it complicates the information revealed from the current interest rate. Instead, the path-dependence is modeled by responding to past beliefs.

[18] In steady state, $E_{t-1}^s r_{t-1}^n = 0$, and $E_{t-1}^s u_{t-1} = 0$.

6 Quantitative Assessment

I now seek to numerically compare the equilibrium interest rate under discretion and the optimal policy rule in the dynamic model. In this section, I adopt parameter values in align with traditional literature, and vary the distribution of external signals to see how the size of the gains from commitment depend on the level of informational frictions in the private sector.

As noted in Section 2, I set $\varphi = 1$ and $\sigma = 1$, assuming a unitary Frisch elasticity of labor supply and log utility of consumption. I use $\beta = 0.99$, which implies a steady state real return on financial assets of four percent. For price rigidity, I calibrate θ, the price stickiness parameter, to be 0.5, which is indicated by the average price duration from macro and micro empirical evidences. [19] For the parameter that governs the elasticity of substitution between intermediate goods, I set $\varepsilon = 4$, which implies a steady state price markup of one-third of revenue.

For the evolution of underlying shocks, I set the auto-correlation of natural-rate shocks to be 0.9, with a standard deviation of 3 percent, as measured by Laubach and Williams (2003). There is less consensus in the persistence and volatility of cost-push shocks, as they stems from a various sources. I set the auto-correlation for cost-push shocks to be 0.3 to avoid informational effect of interest rate being so strong that kills the equilibrium of an optimizing discretionary interest rate. I set the standard deviation of cost-push shocks to be the same as that of natural-rate shocks. In addition, I set the standard deviation of policy implementation error to be the same as the standard deviation of natural rate shock. I set the standard deviations in the noise of the external signals to be 0.1 in the impulse response calculation, and use various numbers when quantify the gains from commitment.

6.1 Impulse Response Analysis

The numerical solution for the dynamic model yields the following result for the equilibrium interest rate under discretion and the optimal policy rule:

$$i_{discretionary} = 1.9658 r_t^n - 0.8692 E_{t-1}^s r_{t-1}^n + 0.3071 u_t + 0.3038 E_{t-1} u_{t-1}, \tag{64}$$

$$i_{rule} = 1.9308 r_t^n - 0.8044 E_{t-1}^s r_{t-1} + 0.0009 u_t + 1.0156 E_{t-1}^s u_{t-1}. \tag{65}$$

The associated equilibrium output gap and inflation are shown in the following impulse response to (a) an natural-rate shock, (b) a cost-push shock, and (c) a policy implementation error.

[19] Sources:Bils and Klenow (2004), Galı and Gertler (1999), Nakamura and Steinsson (2010)

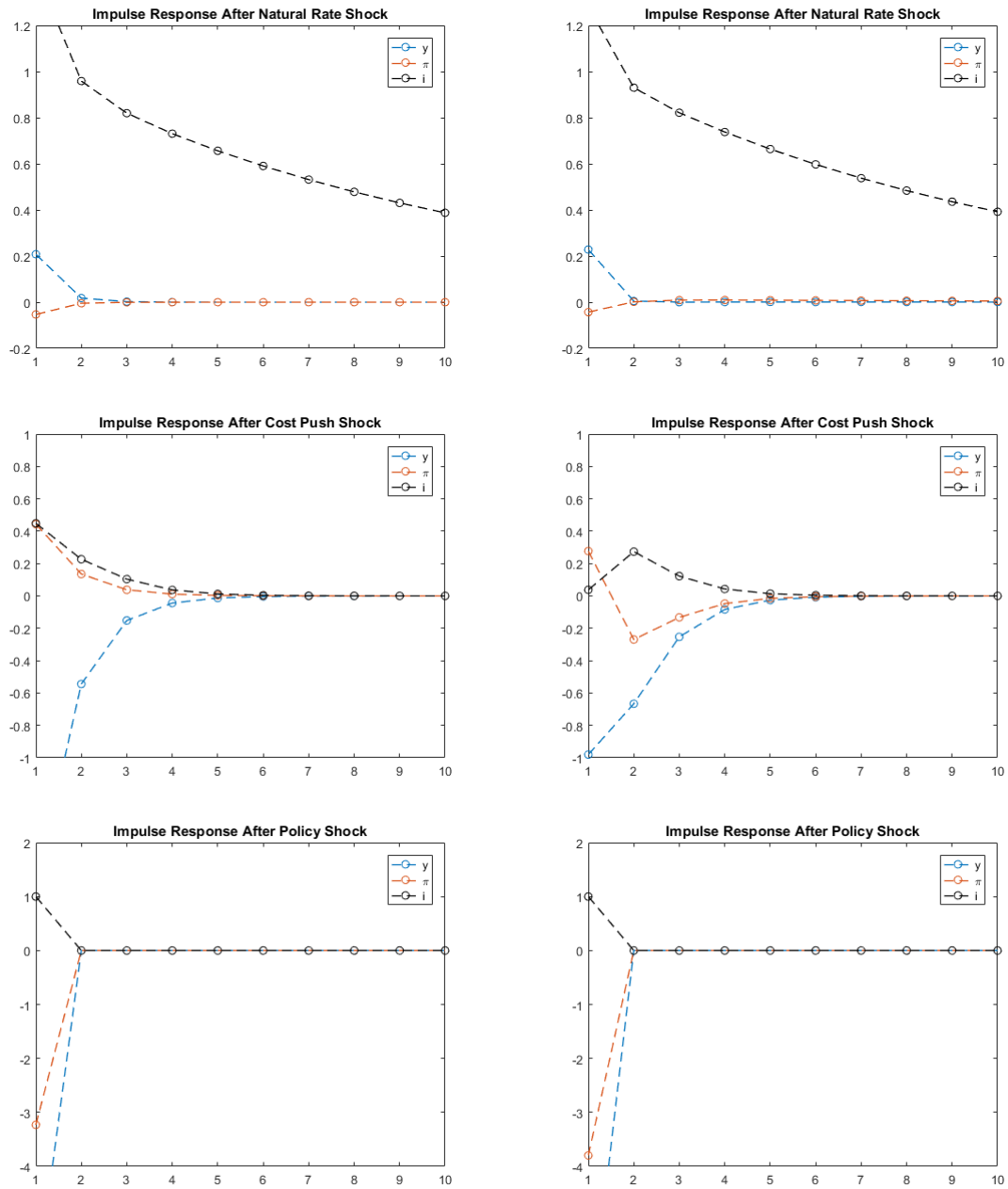

Figure 6: Impulse Response under Discretion (Left) and under Commitment (Right)

First compare the response of the interest rate to a natural rate shock. The coefficients of interest rate to the actual natural-rate shock and to the past expected shock are very similar. The interest rate under both discretion and commitment reacts more than one-to-one to changes in the interest rate, and reacts negatively to the past expected shocks. To understand the intuition, notice that as suggested in the IS curve in equation (10), the serial correlation enlarges the positive output gap caused by r_t^n, and the $E_t^s r_t^n$ reduces the positive output gap. As a result, the interest rate responds

more than one-to-one to r_t^n and responds negatively to $E_t^s r_t^n$.

The response to cost-push shocks are significantly different under discretion and under commitment. In both cases, the interest rate exhibits inertia after cost-push shocks, which is reflected by a positive response to $E_{t-1}^s u_{t-1}$. However, the degree of inertia is much greater under commitment. In the impulse response figure, the interest rate under commitment has a humped shape after a cost-push shock.

After a cost-push shock, by committing to respond less in the first period and committing to a higher interest rate in future periods, the central bank changes the expectations on both the current realization of shocks and on future equilibrium variables. Specifically, the interest rate becomes a more precise signal for the natural-rate shock, and reveals little information about the cost-push shock. Consequently, private agents barely update their expectations about the cost-push shock, which reduces expected inflation and thus actual inflation as well. In addition, the central bank with optimal commitment will also tighten monetary policy in future periods than what is optimal under discretion. As pricing decisions are forward-looking, expectations of a lower future inflation decreases current inflation. In summary, the traditionally studied gains from a delayed response reinforce the informational gains from commitment.

6.2 The Size of the Gains from Commitment

As Section 4 shows, the gains from commitment depend on the interaction between the informational effect of the interest rate and the information conveyed through external signals. I stop short on calibrating the precision of external signals. Instead, I vary the precision of external signals to see the degree to which the size of the gains from commitment depend on it. In the following table, I report the ex-ante loss, the variance of inflation and the variance of the output gap when the external signals change from precise to imprecise.

	Discretionary			Rule		
	$\sigma_\varepsilon = 0.03$	$\sigma_{\varepsilon r} = 1$	$\sigma_\varepsilon = 10$	$\sigma_\varepsilon = 0.03$	$\sigma_{\varepsilon r} = 1$	$\sigma_\varepsilon = 10$
Ex-ante Loss	5.87	3.75	3.74	3.81	3.62	3.60
Inflation	1.67	1.01	1.00	0.76	0.93	0.94
Output Gap	28.64	18.71	18.69	14.46	17.75	17.55

Table 1: Gains from Commitment at Varying Levels of Precision of External Information

The ex-ante loss is calculated as the objective function of the central bank defined in equation (57) $\times 10^2$. The numbers for inflation and output gap are noted in percentage points.

7 Conclusion

In this paper, I studied an economy in which the private sector has imperfect information about the underlying shocks and the central bank has perfect information when making interest rate decisions. Consequently, the interest rate decisions have an informational effect, as the private sector regards the interest rate as a signal about the unobserved shocks, and extracts information from it. I showed that with serially correlated shocks and relatively precise external information, the size of gains from commitment is quantitatively important.

To theoretically study the gains from commitment, I built a New Keynesian model with both nominal frictions and information frictions, and studied the optimal response of interest rates to natural-rate shocks and cost-push shocks. I started with the simple scenario in which both shocks are serially uncorrelated, and imposed the restriction that interest rate could only respond to current shocks. This two assumptions allowed me to isolate the informational gains from commitment.

A discretionary central bank sets interest rates to optimize its objective function at any state of the economy, taking as given the informational effect of its interest rate decision. In comparison, under commitment, a central bank can change the informational effect of interest rates by committing to a different response function than its optimizing response under discretion. Consequently, the informational effect of interest rates makes the Phillips curve endogenous to the central bank's interest-rate decision. By committing to the optimal policy rule, the central bank is able to reduce the degree to which the informational effect dampens the direct effect of the interest rate.

I extend the analysis by adding external signals in addition to the interest rate. External signals can also model central bank direct communication, which interacts with the informational effect of the interest rate. I presented the situation in which providing more precise information about the efficient shocks might reduce welfare. In this case, communicating about the natural-rate shock also makes the interest-rate a more precise signal about the cost-push shock.

Finally, I quantified the size of the gains from commitment by adopting conventionally used parameter values while varying the precision of external signals. I found that when external signals are extremely imprecise, the size of gains from commitment is negligible. However, more precise external information about both shocks increases the size of gains from commitment. Specifically, when the precision of external signals is equal to the prior distribution of actual shocks, committing to the optimal policy rule improves ex-ante welfare by 35 percent relative to the equilibrium under the optimizing discretionary policy.

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Appendices

A Log-Linearization and Aggregation

From the household first order conditions, we first do log-linear approximation to the Euler equation in (A.6) by

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) \tag{A.1}$$

The log-linear approximation to the labor supply of equation (A.7) is $\varphi n_t(j) + \sigma y_t = w_t(j)$ where w_t denotes the log approximated real wage, $log(W_t/P_t)$. Recall that resource constraint implies that $c_t^j = y_t^j \; \forall j$, which further implies $c_t = y_t$. We can then write the labor supply as follows:

$$\varphi n_t(j) + \sigma y_t = w_t(j) \tag{A.2}$$

Next, we want to relate individual firm's real marginal cost of production to aggregate output. To to this, first integrate equation (A.13):

$$\int w_t(j) = \varphi \int n_t(j) dj + \sigma y_t \tag{A.3}$$

Then, substitute the log-linear approximation of the individual good demand, i.e., $y_t(j) - y_t = -\varepsilon(p_t(j) - p_t)$, which results in:

$$\int n_t(j) dj = y_t + \int (-\varepsilon)(p_t(j) - p_t) - \int a_t(j) = y_t - a_t \tag{A.4}$$

Substitute this into $\int w_t(j)$, and then deduct a_t from both sides:

$$\int w_t(j) - a_t(j) = (\phi + \sigma) y_t - (1 + \varphi) a_t \tag{A.5}$$

Define natural level of output as the equilibrium output level without price rigidity and under perfect information, which makes y_t^n as a linear function of aggregate technology. Then, write the above equation in terms of output gap:

$$\int w_t(j) - a_t(j) = (\phi + \sigma)(y_t - y_t^n) \tag{A.6}$$

We know move on to the firm's side. Taking log-linear approximation of individual firm's optimal resetting prices:

$$p_t^*(j) = (1 - \beta\theta) E_t^j \left\{ \Sigma (\beta\theta)^k \left[p_{t+k} + u_{t+k}(j) + w_{t+k}(j) - a_{t+k}(j) \right] \right\} \tag{A.7}$$

The Calvo assumption implies that the aggregate price index is an average of the price charged by the fraction of $1-\theta$ of firms which reset their prices at t, and the fraction of θ of firms whose prices remain as the last period prices. Thus, the log-linear approximation of the aggregate price in period t becomes:

$$p_t = \theta p_{t-1} + (1-\theta)\int p_t^*(j)dj \tag{A.8}$$

Subtract p_{t-1} from both sides to express in terms of inflation:

$$\pi_t = (1-\theta)\left(\int p_t^*(j) - p_{t-1}\right) \tag{A.9}$$

As explained in Section 2.3, assume homogeneous, subjective believes in order to abstract from the higher order beliefs problem in aggregating prices. This assumption allows me to write individual resetting prices as:

$$p_t^*(j) = (1-\beta\theta)\left(E_t^s p_t + u_t(j) + w_t(j) - a_t(j)\right) + (1-\beta\theta)\Sigma_{k=1}^{\infty}(\beta\theta)^k E_t^s\left(p_{t+k} + u_{t+k}(j) + w_{t+k} - a_{t+k}\right) \tag{A.10}$$

Integrate over j:

$$\int p_t^*(j)dj = (1-\beta\theta)\left(E_t^s p_t + u_t + w_t - a_t\right) + (1-\beta\theta)\Sigma_{k=1}^{\infty}(\beta\theta)^k E_t^s\left(p_{t+k} + w_{t+k} - a_{t+k}\right) \tag{A.11}$$

To write in difference equation, first calculate:

$$\beta\theta\int E_t^s p_{t+1}^*(j)dj = (1-\beta\theta)\Sigma_{k=1}^{\infty}E_t^s\left(p_{t+k} + u_{t+k} + w_{t+k} - a_{t+k}\right) = \beta\theta E_t^s p_{t+k}^* \tag{A.12}$$

The second equation holds due to homogeneous beliefs.

Subtract equation (A. 23) from equation (A. 22):

$$\int p_t^*(j)dj - \beta\theta E_t^s p_{t+1} = (1-\beta\theta)E_t^s p_t + (1-\beta\theta)u_t + (1-\beta\theta)(\varphi+\sigma)\hat{y} \tag{A.13}$$

$$\int p_t^*(j)dj - p_{t-1} = \beta\theta\left(E_t^s p_{t+1}^* - E_t^s p_t\right) + E_t^s p_t - p_{t-1} + (1-\beta\theta)u_t + (1-\beta\theta)(\varphi+\sigma)\hat{y}_t$$

$$\pi_t = \beta\theta E_t^s \pi_{t+1} + (1-\theta)E_t^s \pi_t + (1-\theta)(1-\beta\theta)u_t + (1-\beta\theta)(1-\theta)(\varphi+\sigma)\hat{y}_t$$

In the last equation, I assume that aggregate price is observable after one period, i.e., $p_{t-1} = E_t^s p_{t-1}$

Write inflation as:

$$\pi_t = \beta\theta E_t^s \pi_{t+1} + (1-\theta)E_t^s \pi_t + \kappa\theta\hat{y}_t + u_t \tag{A.14}$$

where $\kappa = \frac{(1-\beta\theta)(1-\theta)(\varphi+\sigma)}{\theta}$, and $u_t = (1-\theta)(1-\beta\theta)u_t$

B Solution to the Markov Perfect Equilibrium under Discretionary Monetary Policy

In this section, I first solve the model with serially uncorrelated shocks and then solve the model with serially correlated shocks. For both cases, I solve for the fixed point where the beliefs by people in the private sector on the best response of interest rate at any state match the optimizing discretionary interest rate. This means that in equilibrium people have rational expectation.

B.1 Equilibrium Optimizing Discretionary Policy with Serially Uncorrelated Shocks

Summary of the iteration process:

1. I conjecture that interest rate reacts linear to both shocks, i.e., $i_t = F_r^0 r_t^n + F_u^0 u_t$.
2. With this interest rate, I solve for the beliefs formed about natural-rate shock and cost-push shock in the private sector as functions of i_t.
3. With beliefs formed in private sector, $E_t^s r_t^n$ and $E_t^s u_t$, the actual shocks, r_t^n and u_t, I solve for $\hat{y}_t$ and π_t as a function of i_t, r_t^n and u_t.
4. Solve for i_t that minimizes the loss function, $L_t = \pi_t^2 + \omega \hat{y}_t$, and express interest rate as actual shocks, $i_t = F_r r_t^n + F_u u_t$.
5. Check if $F_r = F_r^0$ and $F_u = F_u^0$. If not, go back to step 1 and update the values of F_r^0 and F_u^0 in the conjectured function. Iterate the process until convergence.

Details are given as follows:
In step 1, $i_t = F_r^0 r_t^n + F_u^0 u_t$.
In step 2, beliefs about underlying shocks follow:

$$E_t^s r_t^n = K_r i_t \tag{B.1}$$

$$E_t^s u_t = K_u i_t \tag{B.2}$$

where $K_r F_r^0 = \frac{F_r^{02}\sigma_r^2}{F_r^{02}\sigma_r^2 + F_u^{02}\sigma_u^2}$, and $K_u F_u^0 = \frac{F_u^{02}\sigma_u^2}{F_r^{02}\sigma_r^2 + F_u^{02}\sigma_u^2}$.

In step 3, write out the expression of output gap and inflation as function of interest rate:

$$\hat{y}_t = -\frac{1}{\sigma}\left(i_t - r_t^n\right) \tag{B.3}$$

$$\pi_t = \kappa \hat{y}_t + (1-\theta)\frac{\kappa}{\sigma}\left(E_t^s r_t^n(i_t) - r_t^n\right) + \frac{1-\theta}{\theta} E_t^s u_t(i_t) + u_t \tag{B.4}$$

In step 4, I first write out the first order condition of interest rate:

$$\pi_t \frac{\partial \pi_t}{\partial i_t} + \omega \hat{y}_t \frac{\partial \hat{y}_t}{\partial i_t} = 0 \tag{B.5}$$

Substitute π_t and $\hat{y}_t$ by equation (B.3) and (B.4):

$$\left\{ (1-\theta)\frac{\kappa}{\sigma}(E_t^s r_t^n - r_t^n) + \frac{1-\theta}{\theta} E_t^s u_t + u_t \right\} \frac{\partial \pi_t}{\partial i_t} + \left(\omega \frac{\partial \hat{y}_t}{\partial i_t} + \kappa \frac{\partial \pi_t}{\partial i_t} \right) \left\{ -\frac{1}{\sigma}(i_t - r_t^n) \right\} = 0 \tag{B.6}$$

Substituting $E_t^s r_t^n$ and $E_t^s u_t$ as i_t leads to:

$$\lambda_1 r_t^n + \lambda_2 u_t + \lambda_3 i_t = 0 \tag{B.7}$$

where

$$\begin{aligned}
\lambda_1 &= \left\{ \left(\kappa \frac{\partial \pi_t}{\partial i_t} + \omega \frac{\partial \hat{y}_t}{\partial i_t} \right) \frac{1}{\sigma} - \frac{\partial \pi_t}{\partial i_t}(1-\theta)\frac{\kappa}{\sigma} \right\} \\
\lambda_2 &= \frac{\partial \pi_t}{\partial i_t} \\
\lambda_3 &= \frac{\partial \pi_t}{\partial i_t}(1-\theta)\frac{\kappa}{\sigma} K_{11} + \frac{\partial \pi_t}{\partial i_t}\frac{1-\theta}{\theta} K_{21} - \left(\kappa \frac{\partial \pi_t}{\partial i_t} + \omega \frac{\partial \hat{y}_t}{\partial i_t} \right) \frac{1}{\sigma}
\end{aligned}$$

and partial derivatives are given by:

$$\begin{aligned}
\frac{\partial \hat{y}_t}{\partial i_t} &= -\frac{1}{\sigma} \\
\frac{\partial \pi_t}{\partial i_t} &= -\frac{\kappa}{\sigma} + (1-\theta)\frac{\kappa}{\sigma} K_r + \frac{1-\theta}{\theta} K_u
\end{aligned}$$

Rearranging the above equation to get:

$$i_t = F_1 r_t^n + F_3 u_t \tag{B.8}$$

where $F_1 = -\frac{\lambda_1}{\lambda_3}$, and $F_3 = -\frac{\lambda_2}{\lambda_3}$.

In step 5, update the initial conjectured policy function and iterate the above process until $F_r = F_r^0$ and $F_u = F_u^0$.

B.2 Equilibrium Optimizing Discretionary Policy with Serially Correlated Shocks

In this section, I solve for the general version of the model where I have serially correlated shocks, external signals which captures central bank direct communication, and implementation error.

Due to the dynamic learning process, expectations on future equilibrium play a role in current decisions. I use the method of undetermined coefficients to solve the subjective expectations formed by private agents. I am then able to express current inflation and the output gap in terms of actual shocks and beliefs in current period. The solution of equilibrium interest rate under discretion takes similar process as the static case, in which I first conjecture the policy function for the interest rate, and then find the fixed point between the initial guess and the interest rate found in the central bank's optimization problem. Summary of the iteration process are given as follows:

1. I conjecture that the interest rate reacts linearly to pre-determined state variables which include current actual shocks and beliefs in last period, i.e., $i_t = F_1^0 r_t^n + F_2^0 E_{t-1}^s r_{t-1}^n + F_3^0 u_t + F_4^0 E_{t-1}^s u_{t-1}$.

2. With this interest rate, I solve for the beliefs about the natural rate and the cost push shock in this period, $E_t^s r_t^n$ and $E_t^s u_t$.

3. (Undetermined Coefficient) I conjecture that output gap and inflation are linear functions of current state variables which include actual shocks and past beliefs. Due to the serial correlation in shocks and the conjectured linear relationship, I am able to express the expected future output gap and inflation as functions of $[E_t^s r_t^n,\ E_t^s u_t]$.

4. I solve for $\hat{y}_t$ and π_t as a function of i_t and other pre-determined state variables.

5. Solve for i_t that minimizes the loss function, $L_t = \pi_t^2 + \omega \hat{y}_t$, and express interest rate as actual shocks, $i_t = F_r r_t^n + F_u u_t$.

6. Check if $F = F_0$. If not, go back to step 1 and update the initial guess of the coefficients in the policy function. Iterate the process until convergence.

Specifically, in step 1, I conjecture that $i_t = F_1 r_t^n + F_r E_{t-1}^s r_{t-1}^n + F_3 u_t + F_4 E_{t-1}^s u_{t-1}$.

In Step 2, to solve the beliefs formed in the pirvate sector, I first specify the evolution of actual shocks:

State:

$$\begin{bmatrix} r_t^n \\ u_t \end{bmatrix} = \begin{bmatrix} \phi & 0 \\ 0 & \phi^u u_t \end{bmatrix} + \begin{bmatrix} v_t \\ v_t^u \end{bmatrix} \tag{B.9}$$

which I denote as $z_t = \Phi z_{t-1} + v_t$, where $\Phi = \begin{bmatrix} \phi & 0 \\ 0 & \phi^u \end{bmatrix}$ and $v_t = [v_t, v_t^u]$ with the white noise of variance Q.

Signals

As private agents have perfect memory of beliefs they have in the past, they are able to back out the part of interest rate that reacts to current shocks, which I denote as

$$\hat{i}_t \equiv i_t - F_2 E^s_{t-1} r^n_{t-1} - F_4 E^s_{t-1} u_{t-1} \tag{B.10}$$

All signals are summarized as.

$$\begin{bmatrix} \hat{i}_t \\ m^r_t \\ m^u_t \end{bmatrix} = \begin{bmatrix} F_1 & F_3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r^n_t \\ u_t \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_t \\ \varepsilon^r_t \\ \varepsilon^u_t \end{bmatrix} \tag{B.11}$$

which I denote as $s_t = Dz_t + R_t$

Beliefs

People in private sector are Bayesian, and update beliefs through the Kalman Filtering process, in which they optimally weigh between all current signals and past beliefs by their precision. The beliefs follow:

$$\begin{bmatrix} E^s_t r^n_t \\ E^s_t u_t \end{bmatrix} = \begin{bmatrix} \phi & 0 \\ 0 & \phi^u \end{bmatrix} \begin{bmatrix} E^s_{t-1} r^n_{t-1} \\ E^s_{t-1} u_{t-1} \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \end{bmatrix} \left(\begin{bmatrix} \hat{i}_t \\ m^r_t \\ m^u_t \end{bmatrix} - \begin{bmatrix} F_1 & F_3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi & 0 \\ 0 & \phi^u \end{bmatrix} \begin{bmatrix} E^s_{t-1} r^n_{t-1} \\ E^s_{t-1} u_{t-1} \end{bmatrix} \right) \tag{B.12}$$

Write out the expression for $\hat{i}_t$ and collect terms:

$$\begin{aligned} E^s_t r^n_t &= (K_{11}F_1 + K_{12})\, r^n_t + \phi\,(1 - K_{11}F_1 - K_{12})\, E^s_{t-1} r^n_{t-1} \\ &\quad + (K_{11}F_3 + K_{13})\, u_t + \phi^u\,(-K_{11}F_3 - K_{13})\, E^s_{t-1} u_{t-1} + K_{12}\varepsilon^r_t + K_{13}\varepsilon^u_t + K_{11} e_t \end{aligned} \tag{B.13}$$

$$\begin{aligned} E^s_t u_t &= (K_{21}F_1 + K_{22})\, r^n_t + \phi\,(-K_{21}F_1 - K_{22})\, E^s_{t-1} r^n_{t-1} \\ &\quad + (K_{21}F_3 + K_{23})\, u_t + \phi^u\,(1 - K_{21}F_3 - K_{23})\, E^s_{t-1} u_{t-1} + K_{22}\varepsilon^r_t + K_{23}\varepsilon^u_t + K_{21} e_t \end{aligned} \tag{B.14}$$

Denote the above equations as $E^s_t r^n_t = \Psi(1) r^n_t + \Psi(2) E^s_{t-1} r^n_{t-1} + \Psi(3) u_t + \Psi(4) E^s_{t-1} u_{t-1} + \Psi(5)\varepsilon^r_t + \Psi(6)\varepsilon^u_t + \Psi(7) e_t$, and $E^s_t u_t = \Psi(8) r^n_t + \Psi(9) E^s_{t-1} r^n_{t-1} + \Psi(10) u_t + \Psi(11) E^s_{t-1} u_{t-1} + \Psi(12)\varepsilon^r_t + \Psi(13)\varepsilon^u_t + \Psi(14) e_t$. I will use this notation in solving equilibrium in the private sector by the method of undetermined coefficients.

In step 3, the first write out the the forward-looking output gap and inflation as:

$$\hat{y}_t = E^s_t \hat{y}_{t+1} - \frac{1}{\sigma}\left[i_t - \left(\frac{1}{1-\phi} r^n_t - \frac{\phi}{1-\phi} E^s_t r^n_t \right) - E^s_t \pi_{t+1} \right] \tag{B.15}$$

$$\pi_t = \beta\theta E^s_t \pi_{t+1} + (1-\theta) E^s_t \pi_t + \kappa\theta \hat{y}_t + u_t \tag{B.16}$$

Following the method of undetermined coefficients, I first need to conjecture that equilibrium variables are linear functions to current state variables, which include current actual shocks $(r_t^n,\ u_t)$, past beliefs, $\left(E_{t-1}^s r_{t-1}^n,\ E_{t-1}^s u_{t-1}\right)$, and noise in current signals, $(\varepsilon_t^r,\ \varepsilon_t^u,\ e_t)$.

$$\begin{bmatrix} \hat{y}_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} \end{bmatrix} \begin{bmatrix} r_t^n \\ E_{t-1}^s r_{t-1}^n \\ u_t \\ E_{t-1}^s u_{t-1} \end{bmatrix} + \begin{bmatrix} \gamma_5 & \gamma_6 & \gamma_7 \\ \gamma_{12} & \gamma_{13} & \gamma_{14} \end{bmatrix} \begin{bmatrix} \varepsilon_t^r \\ \varepsilon_t^u \\ e_t \end{bmatrix} \tag{B.17}$$

Next, substitute this conjecture into the forward-looking variables, $E_t\hat{y}_{t+1}$ and $E_t^s\pi_{t+1}$. Notice that noise of all signals are temporary, which are expected to be zero in future period.

$$\begin{bmatrix} E_t^s\hat{y}_{t+1} \\ E_t^s\pi_{t+1} \end{bmatrix} = \begin{bmatrix} \gamma_1\phi + \gamma_2 & \gamma_3\phi^u + \gamma_4 \\ \gamma_8\phi + \gamma_9 & \gamma_{10}\phi^u + \gamma_{11} \end{bmatrix} \begin{bmatrix} E_t^s r_t^n \\ E_t^s u_t \end{bmatrix} \tag{B.18}$$

First substitute this into the output gap expression:

$$\hat{y}_t = \left[(\gamma_1\phi + \gamma_2) + \frac{1}{\sigma}(\gamma_8\phi + \gamma_9) - \frac{1}{\sigma}\frac{\phi}{1-\phi}\right] E_t^s r_t^n \tag{B.19}$$

$$+ \left[(\gamma_3\phi^u + \gamma_4) + \frac{1}{\sigma}(\gamma_{10}\phi^u + \gamma_{11})\right] E_t^s u_t - \frac{1}{\sigma} i_t + \frac{1}{\sigma}\frac{1}{1-\phi} r_t^n \tag{B.20}$$

Next work on π_t, as the actual inflation also includes the expected current inflation, and expected current inflation includes expected current output gap, I first need to calculate:

$$E_t^s\hat{y}_t = E_t\hat{y}_{t+1} - \frac{1}{\sigma}\left[i_t - E_t^s r_t^n - E_t^s\pi_{t+1}\right] \tag{B.21}$$

$$E_t^s\pi_t = \beta E_t^s\pi_{t+1} + \kappa\left\{E_t^s\hat{y}_{t+1} - \frac{1}{\sigma}\left[i_t - E_t^s r_t^n - E_t^s\pi_{t+1}\right]\right\} + \frac{1}{\theta}E_t^s u_t \tag{B.22}$$

Substitute $E_t^s\pi_t$ into π_t:

$$\pi_t = \beta\theta E_t^s\pi_{t+1} + (1-\theta)\left\{\beta E_t^s\pi_{t+1} + \kappa E_t^s\hat{y}_t + \frac{1}{\theta}E_t^s u_t\right\} + \kappa\theta\hat{y}_t + u_t \tag{B.23}$$

$$+ \left\{(1-\theta)\kappa(\gamma_3\phi^u + \gamma_4) + \frac{1-\theta}{\theta} + \left(\beta + (1-\theta)\frac{\kappa}{\sigma}\right)(\gamma_{10}\phi^u + \gamma_{11})\right\} E_t^s u_t - (1-\theta)\frac{\kappa}{\sigma} i_t + \kappa\theta\hat{y}_t + u_t$$

The values of γ can be solved in the following matrix:

$$\gamma = M\gamma + c$$

where

$$\gamma = [\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \gamma_6,\ \gamma_7,\ \gamma_8]'$$

$$M = \begin{bmatrix}
\phi\Psi_1 & \Psi_1 & \phi^u\Psi_8 & \Psi_8 & 0 & 0 & 0 & \frac{1}{\sigma}\phi\Psi_1 & \frac{1}{\sigma}\Psi_1 & \frac{1}{\sigma}\phi^u\Psi_8 & \frac{1}{\sigma}\Psi_8 & 0 & 0 \\
\phi\Psi_2 & \Psi_2 & \phi^u\Psi_9 & \Psi_9 & 0 & 0 & 0 & \frac{1}{\sigma}\phi\Psi_2 & \frac{1}{\sigma}\Psi_2 & \frac{1}{\sigma}\phi^u\Psi_9 & \frac{1}{\sigma}\Psi_9 & 0 & 0 \\
\phi\Psi_3 & \Psi_3 & \phi^u\Psi_{10} & \Psi_{10} & 0 & 0 & 0 & \frac{1}{\sigma}\phi\Psi_3 & \frac{1}{\sigma}\Psi_3 & \frac{1}{\sigma}\phi^u\Psi_{10} & \frac{1}{\sigma}\Psi_{10} & 0 & 0 \\
\phi\Psi_4 & \Psi_4 & \phi^u\Psi_{11} & \Psi_{11} & 0 & 0 & 0 & \frac{1}{\sigma}\phi\Psi_4 & \frac{1}{\sigma}\Psi_4 & \frac{1}{\sigma}\phi^u\Psi_{11} & \frac{1}{\sigma}\Psi_{11} & 0 & 0 \\
\phi\Psi_5 & \Psi_5 & \phi^u\Psi_{12} & \Psi_{12} & 0 & 0 & 0 & \frac{1}{\sigma}\phi\Psi_5 & \frac{1}{\sigma}\Psi_5 & \frac{1}{\sigma}\phi^u\Psi_{12} & \frac{1}{\sigma}\Psi_{12} & 0 & 0 \\
\phi\Psi_6 & \Psi_6 & \phi^u\Psi_{13} & \Psi_{13} & 0 & 0 & 0 & \frac{1}{\sigma}\phi\Psi_6 & \frac{1}{\sigma}\Psi_6 & \frac{1}{\sigma}\phi^u\Psi_{13} & \frac{1}{\sigma}\Psi_{13} & 0 & 0 \\
\phi\Psi_7 & \Psi_7 & \phi^u\Psi_{14} & \Psi_{14} & 0 & 0 & 0 & \frac{1}{\sigma}\phi\Psi_7 & \frac{1}{\sigma}\Psi_7 & \frac{1}{\sigma}\phi^u\Psi_{14} & \frac{1}{\sigma}\Psi_{14} & 0 & 0 \\
(1-\theta)\kappa\phi\Psi_1+\kappa\theta & (1-\theta)\kappa\Psi_1 & (1-\theta)\kappa\phi^u\Psi_8 & (1-\theta)\kappa\Psi_8 & 0 & 0 & 0 & \left((1-\theta)\frac{\kappa}{\sigma}\phi+\beta\phi\right)\Psi_1 & \left((1-\theta)\frac{\kappa}{\sigma}+\beta\right)\Psi_1 & \left((1-\theta)\frac{\kappa}{\sigma}\phi^u+\beta\phi^u\right)\Psi_8 & \left((1-\theta)\frac{\kappa}{\sigma}+\beta\right)\Psi_8 & 0 & 0 \\
(1-\theta)\kappa\phi\Psi_2 & (1-\theta)\kappa\Psi_2+\kappa\theta & (1-\theta)\kappa\phi^u\Psi_8 & (1-\theta)\kappa\Psi_8 & 0 & 0 & 0 & \left((1-\theta)\frac{\kappa}{\sigma}\phi+\beta\phi\right)\Psi_2 & \left((1-\theta)\frac{\kappa}{\sigma}+\beta\right)\Psi_2 & \left((1-\theta)\frac{\kappa}{\sigma}\phi^u+\beta\phi^u\right)\Psi_9 & \left((1-\theta)\frac{\kappa}{\sigma}+\beta\right)\Psi_9 & 0 & 0 \\
(1-\theta)\kappa\phi\Psi_3 & (1-\theta)\kappa\Psi_3 & (1-\theta)\kappa\phi^u\Psi_{10}+\kappa\theta & (1-\theta)\kappa\Psi_{10} & 0 & 0 & 0 & \left((1-\theta)\frac{\kappa}{\sigma}\phi+\beta\phi\right)\Psi_3 & \left((1-\theta)\frac{\kappa}{\sigma}+\beta\right)\Psi_3 & \left((1-\theta)\frac{\kappa}{\sigma}\phi^u+\beta\phi^u\right)\Psi_{10} & \left((1-\theta)\frac{\kappa}{\sigma}+\beta\right)\Psi_{10} & 0 & 0 \\
(1-\theta)\kappa\phi\Psi_4 & (1-\theta)\kappa\Psi_4 & (1-\theta)\kappa\phi^u\Psi_{11} & (1-\theta)\kappa\Psi_{11}+\kappa\theta & 0 & 0 & 0 & \left((1-\theta)\frac{\kappa}{\sigma}\phi+\beta\phi\right)\Psi_4 & \left((1-\theta)\frac{\kappa}{\sigma}+\beta\right)\Psi_4 & \left((1-\theta)\frac{\kappa}{\sigma}\phi^u+\beta\phi^u\right)\Psi_{11} & \left((1-\theta)\frac{\kappa}{\sigma}+\beta\right)\Psi_{11} & 0 & 0 \\
(1-\theta)\kappa\phi\Psi_5 & (1-\theta)\kappa\Psi_5 & (1-\theta)\kappa\phi^u\Psi_{12} & (1-\theta)\kappa\Psi_{12} & +\kappa\theta & 0 & 0 & \left((1-\theta)\frac{\kappa}{\sigma}\phi+\beta\phi\right)\Psi_5 & \left((1-\theta)\frac{\kappa}{\sigma}+\beta\right)\Psi_5 & \left((1-\theta)\frac{\kappa}{\sigma}\phi^u+\beta\phi^u\right)\Psi_{12} & \left((1-\theta)\frac{\kappa}{\sigma}+\beta\right)\Psi_{12} & 0 & 0 \\
(1-\theta)\kappa\phi\Psi_6 & (1-\theta)\kappa\Psi_6 & (1-\theta)\kappa\phi^u\Psi_{13} & (1-\theta)\kappa\Psi_{13} & 0 & +\kappa\theta & 0 & \left((1-\theta)\frac{\kappa}{\sigma}\phi+\beta\phi\right)\Psi_6 & \left((1-\theta)\frac{\kappa}{\sigma}+\beta\right)\Psi_6 & \left((1-\theta)\frac{\kappa}{\sigma}\phi^u+\beta\phi^u\right)\Psi_{13} & \left((1-\theta)\frac{\kappa}{\sigma}+\beta\right)\Psi_{13} & 0 & 0 \\
(1-\theta)\kappa\phi\Psi_7 & (1-\theta)\kappa\Psi_7 & (1-\theta)\kappa\phi^u\Psi_{14} & (1-\theta)\kappa\Psi_{14} & 0 & 0 & +\kappa\theta & \left((1-\theta)\frac{\kappa}{\sigma}\phi+\beta\phi\right)\Psi_7 & \left((1-\theta)\frac{\kappa}{\sigma}+\beta\right)\Psi_7 & \left((1-\theta)\frac{\kappa}{\sigma}\phi^u+\beta\phi^u\right)\Psi_{14} & \left((1-\theta)\frac{\kappa}{\sigma}+\beta\right)\Psi_{14} & 0 & 0
\end{bmatrix}$$

$$c = \begin{bmatrix}
-\frac{1}{\sigma}\frac{\phi}{1-\phi}\Psi_1 - \frac{1}{\sigma}F_1 + \frac{1}{\sigma}\frac{1}{1-\phi} \\
-\frac{1}{\sigma}\frac{\phi}{1-\phi}\Psi_2 - \frac{1}{\sigma}F_2 \\
-\frac{1}{\sigma}\frac{\phi}{1-\phi}\Psi_3 - \frac{1}{\sigma}F_3 \\
-\frac{1}{\sigma}\frac{\phi}{1-\phi}\Psi_4 - \frac{1}{\sigma}F_4 \\
-\frac{1}{\sigma}\frac{\phi}{1-\phi}\Psi_5 \\
-\frac{1}{\sigma}\frac{\phi}{1-\phi}\Psi_6 \\
-\frac{1}{\sigma}\frac{\phi}{1-\phi}\Psi_7 - \frac{1}{\sigma} \\
(1-\theta)\frac{\kappa}{\sigma}\Psi_1 + \frac{1-\theta}{\theta}\Psi_8 - (1-\theta)\frac{\kappa}{\sigma}F_1 \\
(1-\theta)\frac{\kappa}{\sigma}\Psi_2 + \frac{1-\theta}{\theta}\Psi_9 - (1-\theta)\frac{\kappa}{\sigma}F_2 \\
(1-\theta)\frac{\kappa}{\sigma}\Psi_3 + \frac{1-\theta}{\theta}\Psi_{10} - (1-\theta)\frac{\kappa}{\sigma}F_3 + 1 \\
(1-\theta)\frac{\kappa}{\sigma}\Psi_4 + \frac{1-\theta}{\theta}\Psi_{11} - (1-\theta)\frac{\kappa}{\sigma}F_4 \\
(1-\theta)\frac{\kappa}{\sigma}\Psi_5 + \frac{1-\theta}{\theta}\Psi_{12} \\
(1-\theta)\frac{\kappa}{\sigma}\Psi_6 + \frac{1-\theta}{\theta}\Psi_{13} \\
(1-\theta)\frac{\kappa}{\sigma}\Psi_7 + \frac{1-\theta}{\theta}\Psi_{14} - (1-\theta)\frac{\kappa}{\sigma}
\end{bmatrix}$$

γ can be uniquely pinned down by the above linear system.

In step 5, in order to solve for the optimizing interest rate, I first need to specify central bank's objective function.

Central Bank Objective Function

As current interest rate has persistent effect through the dynamic learning process, central bank also considers how current interest rate affect future equilibrium. Consequently, the loss function includes output gap and inflation of current and all future periods.

$$E_t L(t) = [\pi_t^2 + \omega \hat{y}_t^2] + \beta E_t(L(t+1)) \tag{B.24}$$

where the $E_t(L(t+1))$ is:

$$\Sigma_{j=1}^{\infty} \beta^j E_t \left\{ \begin{bmatrix} \pi_{t+1} & \hat{y}_{t+j} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} \pi_{t+j} \\ \hat{y}_{t+j} \end{bmatrix} \right\} = \Sigma_{j=1}^{\infty} \beta^j \left\{ \begin{bmatrix} E_t \pi_{t+1} & E_t \hat{y}_{t+j} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} E_t \pi_{t+j} \\ E_t \hat{y}_{t+j} \end{bmatrix} + indept.\, terms \right\} \tag{B.25}$$

The central banks expectation is *objective*, denoted by E_t, in the sense that it observes all past shocks, and expects all future shocks to be zero. The information set of central bank at period t is:

$$I_t = \{r_T^n,\ u_T,\ \forall T = 0...t\}$$

Let $z_t = \left[r_t^n,\ E_{t-1}^s r_{t-1}^n, u_t,\ E_{t-1}^s u_{t-1}\right]'$ denote the persistent state variables. So the central bank's objective expectation of future period output gap and inflation becomes a linear function of $E_t z_{t+j}$:

$$\begin{bmatrix} E_t \pi_{t+j} \\ E_t \hat{y}_{t+j} \end{bmatrix} = \begin{bmatrix} \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix} E_t z_{t+j} \equiv \Gamma E_t z_{t+j} \tag{B.26}$$

$E_t z_{t+j}$ follows:

$$\begin{bmatrix} E_t r_{t+j}^n \\ E_t E_{t+j-1}^s r_{t+j-1}^n \\ E_t u_{t+j} \\ E_t E_{t+j-1}^s u_{t+j-1} \end{bmatrix} = \begin{bmatrix} \phi & 0 & 0 & 0 \\ K_{11}F_1 + K_{12} & \phi(1 - K_{11}F_1 - K_{12}) & K_{11}F_3 + K_{13} & -\phi^u(K_{11}F_3 + K_{13}) \\ 0 & 0 & \phi^u & 0 \\ K_{21}F_1 + K_{22} & -\phi(K_{21}F_1 + K_{22}) & K_{21}F_3 + K_{23}) & \phi^u(1 - K_{21}F_3 - K_{23}) \end{bmatrix} \begin{bmatrix} E_t r_{t+j-1}^n \\ E_t E_{t+j-2}^s r_{t+j-2}^n \\ E_t u_{t+j-1} \\ E_t E_{t+j-2}^s u_{t+j-2} \end{bmatrix} \tag{B.27}$$

$$\begin{bmatrix} E_t \pi_{t+j} \\ E_t \hat{y}_{t+j} \end{bmatrix} = \Gamma \Lambda^{j-1} E_t z_{t+1} \tag{B.28}$$

Substitute into the $E_t(L(t+1))$:

$$\Sigma \beta^j E_t z_{t+1}' (\Lambda^{j-1})' \Gamma' \Omega \Gamma \Lambda^{j-1} E_t z_{t+1} \equiv \Sigma \beta^j E_t z_{t+1}' \Theta_{j-1} E_t z_{t+1} \tag{B.29}$$

Take the first order condition on i_t^* of $E_t L(t+1)$:

$$\left\{ \frac{\partial E_t \pi_t}{\partial i_t^*} E_t \pi_t + \omega \frac{\partial E_t \hat{y}_t}{\partial i_t^*} E_t \hat{y}_t \right\} + \frac{1}{2} \Sigma_{j=1}^{\infty} \beta^j \Delta(j-1) = 0 \tag{B.30}$$

where

$$\begin{aligned}
\Delta_{j-1} &= (\Theta_{j-1}^{21} + \Theta_{j-1}^{12}) \phi r_t^n \frac{\partial E_t^s r_t^n}{\partial i_t} + (\Theta_{j-1}^{32} + \Theta_{j-1}^{23}) \phi^u u_t \frac{\partial E_t^s r_t^n}{\partial i_t} + (\Theta_{j-1}^{42} + \Theta_{j-1}^{24}) E_t^s u_t \frac{\partial E_t^s r_t^n}{\partial i_t} + \Theta_{j-1}^{22} \cdot 2 E_t^s r_t^n \frac{\partial E_t^s r_t^n}{\partial i_t} \\
&+ (\theta_{j-1}^{41} + \Theta_{j-1}^{14}) \phi r_t^n \frac{\partial E_t^s u_t}{\partial i_t} + (\Theta_{j-1}^{43} + \Theta_{j-1}^{34}) \phi^u u_t \frac{\partial E_t^s u_t}{\partial i_t} + (\Theta_{j-1}^{42} + \Theta_{j-1}^{24}) E_t^s r_t^n \frac{\partial E_t^s u_t}{\partial i_t} + \Theta_{j-1}^{44} \cdot 2 E_t^s u_t \frac{\partial E_t^s u_t}{\partial i_t} \\
&\equiv \Delta_{j-1}(1) r_t^n + \Delta_{j-1}(2) u_t + \Delta_{j-1}(3) E_t^s u_t + \Delta_{j-1}(4) E_t^s r_t^n \\
&+ \Delta_{j-1}(5) r_t^n + \Delta_{j-1}(6) u_t + \Delta_{j-1}(7) E_t^s r_t^n + \Delta_{j-1}(8) E_t^s u_t
\end{aligned}$$

To solve for the first order condition on interest rate, first write equilibrium variables in terms of i_t:

Beliefs:

$$\begin{aligned}
E_t^s r_t^n &= \left(\phi(1 - K_{11}F_1 - K_{12}) - K_{11}F_2\right) E_{t-1}^s r_{t-1}^n - \left(K_{11}F_4 + \phi^u(K_{11}F_3 + K_{13})\right) E_{t-1}^s u_{t-1} && \text{(B.31)} \\
&+ K_{12} r_t^n + K_{13} u_t + K_{11} i_t \\
E_t^s u_t &= \left(\phi^u(1 - K_{21}F_3 - K_{23}) - K_{21}F_4\right) E_{t-1}^s u_{t-1} - \left(\phi(K_{21}F_1 + K_{22}) + K_{21}F_2\right) E_{t-1}^s r_{t-1}^n && \text{(B.32)} \\
&+ K_{22} r_t^n + K_{23} u_t + K_{21} i_t && \text{(B.33)}
\end{aligned}$$

Output gap:

$$\hat{y}_t = \Xi(1) E_t^s r_t^n + \Xi(2) E_t^s u_t - \frac{1}{\sigma} i_t + \frac{1}{\sigma} \frac{1}{1-\phi} r_t^n \tag{B.34}$$

Inflation:

$$\pi_t = \kappa \theta \hat{y}_t + \Xi(3) E_t^s r_t^n + \Xi(4) E_t^s u_t - (1-\theta) \frac{\kappa}{\sigma} i_t + u_t \tag{B.35}$$

Substitute the above endogenous variables into the first order condition on i_t:

$$\lambda_1 E_t^s r_t^n + \lambda_2 E_t^s u_t + \lambda_3 r_t^n + \lambda_4 u_t + \lambda_5 i_t = 0 \tag{B.36}$$

where

$$\lambda_1 = \left(\kappa\theta\frac{\partial\pi_t}{\partial i_t} + \omega\frac{\partial\hat{y}_t}{\partial i_t}\right)\Xi(1) + \frac{\partial\pi_t}{\partial i_t}\Xi(3) + \frac{1}{2}\Sigma\beta^j\left(\Delta_{j-1}(4) + \Delta(7)\right) \tag{B.37}$$

$$\lambda_2 = \left(\kappa\theta\frac{\partial\pi_t}{\partial i_t} + \omega\frac{\partial\hat{y}_t}{\partial i_t}\right)\Xi(2) + \frac{\partial\pi_t}{\partial i_t}\Xi(4) + \frac{1}{2}\Sigma\beta^j\left(\Delta_{j-1}(3) + \Delta(8)\right) \tag{B.38}$$

$$\lambda_3 = \left(\kappa\theta\frac{\partial\pi_t}{\partial i_t} + \omega\frac{\partial\hat{y}_t}{\partial i_t}\right)\frac{1}{\sigma}\frac{1}{1-\phi} + \frac{1}{2}\Sigma\beta^j\left(\Delta_{j-1}(1) + \Delta(5)\right) \tag{B.39}$$

$$\lambda_4 = \frac{\partial\pi_t}{\partial i_t} + \frac{1}{2}\Sigma\beta^j\left(\Delta(2) + \Delta(6)\right) \tag{B.40}$$

$$\lambda_5 = \left(\kappa\theta\frac{\partial\pi_t}{\partial i_t} + \omega\frac{\partial\hat{y}_t}{\partial i_t}\right)\left(-\frac{1}{\sigma}\right) + \frac{\partial\pi_t}{\partial i_t}\left(-(1-\theta)\frac{\kappa}{\theta}\frac{1}{\sigma}\right) \tag{B.41}$$

and partial derivatives are derived as:

$$\frac{\partial E_t^s r_t^n}{\partial i_t} = K_{11} \tag{B.42}$$

$$\frac{\partial E_t^s u_t}{\partial i_t} = K_{21} \tag{B.43}$$

$$\frac{\partial\hat{y}_t}{\partial i_t} = \Xi(1)\frac{\partial E_t^s r_t^n}{\partial i_t} + \Xi(2)\frac{\partial E_t^s u_t}{\partial i_t} - \frac{1}{\sigma} \tag{B.44}$$

$$\frac{\partial\pi_t}{\partial i_t} = \kappa\theta\frac{\partial\hat{y}_t}{\partial i_t} + \Xi(3)\frac{\partial E_t^s r_t^n}{\partial i_t} + \Xi(4)\frac{\partial E_t^s u_t}{\partial i_t} - (1-\theta)\frac{\kappa}{\sigma} \tag{B.45}$$

Further substitute $E_t^s r_t^n$ and $E_t^s u_t$:

$$\begin{aligned}
0 = {} & \lambda_1\left\{\left(\phi(1-K_{11}F_1-K_{12}) - K_{11}F_2\right)E_{t-1}^s r_{t-1}^n - \left(K_{11}F_4 + \phi^u(K_{11}F_3+K_{13})\right)E_{t-1}^s u_{t-1} + K_{12}r_t^n + K_{13}u_t + K_{11}i_t\right\} \\
& + \lambda_2\left\{\left(\phi^u(1-K_{21}F_3-K_{23}) - K_{21}F_4\right)E_{t-1}^s u_{t-1} - \left(\phi(K_{21}F_1+K_{22}) + K_{21}F_2\right)E_{t-1}^s r_{t-1}^n + K_{22}r_t^n + K_{23}u_t + K_{21}i_t\right\} \\
& + \lambda_3 r_t^n + \lambda_4 u_t + \lambda_5 i_t
\end{aligned}$$

The above equation solves the optimal nominal interest rate. Comparing with the guessed form yields the solution of $[F_1,\ F_2,\ F_3,\ F_4]$

$$F_1 = -\frac{\lambda_1 K_{12} + \lambda_2 K_{22} + \lambda_3}{\lambda_1 K_{11} + \lambda_2 K_{21} + \lambda_5} \tag{B.46}$$

$$F_2 = -\frac{\lambda_1 \left(\phi(1 - K_{11}F_1 - K_{12}) - K_{12}F_2\right) - \lambda_2 \left(\phi(K_{21}F_1 + K_{22}) + K_{21}F_2\right)}{\lambda_1 K_{11} + \lambda_2 K_{21} + \lambda_5} \tag{B.47}$$

$$F_3 = -\frac{\lambda_1 K_{13} + \lambda_2 K_{23} + \lambda_4}{\lambda_1 K_{11} + \lambda_2 K_{21} + \lambda_5} \tag{B.48}$$

$$F_4 = -\frac{-\lambda_1 \left(K_{11}F_4 + \phi^u(K_{11}F_3 + K_{13})\right) + \lambda_2 \left(\phi^u(1 - K_{21}F_3 - K_{23}) - K_{21}F_4\right)}{\lambda_1 K_{11} + \lambda_2 K_{21} + \lambda_5} \tag{B.49}$$

I iterate the process until the conjectured interest rate function matches the above solution.

C Proofs

Second Order Approximation of Household's Utility Function

Follow Woodford (2003), Gali (2010), Walsh (2010) to prove that maximizing the utility of household is equivalent, up to second order approximation, to

$$W = -\frac{1}{2}E_0\Sigma\beta^t\left((\varepsilon^{-1}+\varphi)\varepsilon^2 var_j(p_t(j)) + (\sigma+\varphi)\hat{y}_t^2\right) \tag{C.1}$$

The next step is to prove the relationship between $var_j(p_t(j))$ with $var(\pi_t)$. Denote $\Delta_t = var_j[logp_{jt}]$. Since $var_j\bar{P}_{t-1} = 0$, we have

$$\begin{aligned}\Delta_t &= var_j[logp_{jt} - \bar{P}_{t-1}] \\ &= E_j[logp_{jt} - \bar{P}_{t-1}]^2 - [E_j logp_{jt} - \bar{P}_{t-1}]^2 \\ &= E_j[logp_{jt-1} - \bar{P}_{t-1}]^2 + (1-\theta)(\int p_{tj}^* - \bar{P}_{t-1})^2 - (\bar{P}_t - \bar{P}_{t-1})^2\end{aligned} \tag{C.2}$$

As noted in Appendix A, $\bar{P}_t = (1-\theta)\int p_{tj}^* + \theta\bar{P}_{t-1}$, we have $(1-\theta)\int logp_{tj}^* + \theta\bar{p}_{t-1}$, which implies that $(1-\theta)\int logp_{tj}^* - (1-\theta)p_{t-1} = \bar{p} - \bar{p}_{t-1}$. So, we have:

$$\int logp_{tj}^* = \left(\frac{1}{1-\theta}\right)(\bar{p}_t - \bar{p}_{t-1}) \tag{C.3}$$

Substitute this into (D.2) and get $\Delta_t = \theta\omega_{t-1} + \left(\frac{\theta}{1-\theta}\right)(\bar{p} - \bar{p}_{t-1})^2$. Applying the definition of inflation results in:

$$E_t\Sigma\beta^t\Delta_t = \frac{\theta}{(1-\theta)(1-\theta\beta)}E_t\Sigma\beta^t\pi_t^2 + t.i.p. \tag{C.4}$$